WISKUNDE OPKNAPKURSUS/

MATHEMATICS REFRESHER COURSE

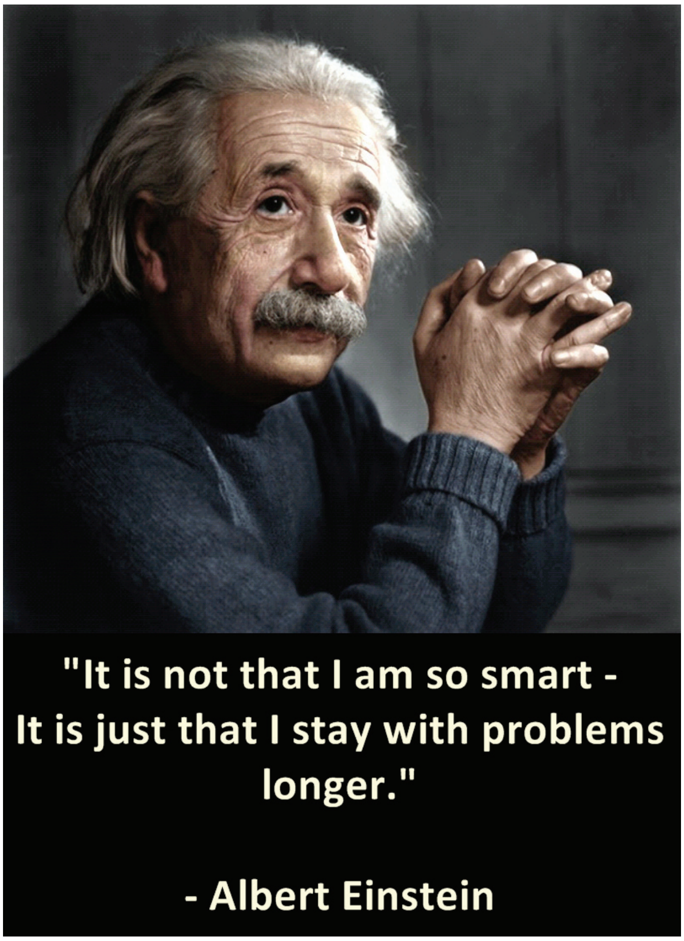
NOTAS EN WERKBOEK VIR/

NOTES AND WORKBOOK FOR

2016

NOORDWES UNIVERSITEIT: POTCHEFSTROOMKAMPUS/

NORTH-WEST UNIVERSITY: POTCHEFSTROOM CAMPUS



NWU®

Materiaal saamgestel deur

Vakgroep Wiskunde van die Fakulteit Natuurwetenskappe

Vakgroep Wiskunde-onderwys van die Fakulteit Opvoedingswetenskappe

Bladuitleg deur Rudi van de Venter, Vakgroep Wiskunde-onderwys

Hersieningsdatum 2016

Noordwes-Universiteit, Potchefstroomkampus

Material compiled by:

Subject Group Mathematics of the Faculty of Natural Sciences

Subject Group Mathematics Education of the Faculty of Education Sciences

Page layout by *Rudi van de Venter, Subject Group Mathematics Education*

Revision date 2016

North-West University, Potchefstroom Campus

Inhoud/ *Contents*

0 Algemene inligting/ *General information*

0.1 Verwelkoming/ *Welcome*

Ons wens u as voornemende Wiskunde-student geluk met u keuse om universiteitstudies aan te pak.

Dit is 'n voorreg om u te ontmoet en 'n rol te mag speel by u voorbereiding op u studies.

Ons as akademiese personeel hoop dat u studies van die begin af voorspoedig en suksesvol sal verloop.

We congratulate you as prospective Mathematics student with your decision to commence university studies.

It is a privilege to meet you and to play a part in your preparation for your studies.

We as academic staff hope that your studies will right from the beginning proceed expediently and successfully.

0.2 Die doel van hierdie kursus/ *Purpose of this course*

Hierdie kursus is ontwerp om:

- U voorkennis van Wiskunde te aktiveer
- U brein se abstrakte en logiese werking na die vakansie aan die gang te kry
- Vae kolle in u wiskunde-kennis en wiskunde-vaardighede op te vul
- U met noodsaaklike kennis en vaardighede waarmee u miskien nie voorheen kennis gemaak het nie, toe te rus
- U bekend te stel aan die werkwyse wat tydens die onderrig van universiteitswiskunde-modules gevolg word
- Die kunsmatige afskortings tussen verskillende "gedeeltes" van Wiskunde af te breek en die verbande tussen elke "deeltjie" en die groot geheel aan te toon, asook die verbande tussen Wiskunde en ander vakdissiplines

This course is designed to:

- *Activate your previous knowledge regarding Mathematics*
- *Jump-start the abstract and logical operation of your brain after the holidays*
- *Fill in hazy spots in your mathematical knowledge*
- *Equip you with essential knowledge and skills with which you possibly have not become aquainted before*
- *Introduce you to the methodogy followed during the teaching of university modules*
- *Break down the artificial compartments between the different "parts" or "sections" of Mathematics and to reveal the relationships between each "part" and the whole, as well as the relationships between Mathematics and other subject disciplines*
- *Alter your experience of Mathematics*

- U belewenis van Wiskunde as vakdissipline te verander sodat dit vir u meer betekenis en lewe as ooit voorheen sal kry

as subject discipline in order for it to acquire more meaning and life for you than ever before

0.3 Studiemateriaal/ *Study material*

In die ou dae het die studiemateriaal vir enige Wiskunde-kursus uit een of meer handboek en klasaantekeninge bestaan, aangevul deur addisionele materiaal vanuit biblioteke. Die prentjie is nou egter anders; u studiemateriaal vir enige moderne Wiskunde-kursus sal waarskynlik 'n kombinasie wees van die volgende:

- Klasaantekeninge (nota's)
- een of meer handboeke
- leermateriaal wat vanaf betroubare internetbronne verkry is
- draagbare sakrekenaar (met of sonder die vermoë om grafieke en tabelle te genereer)
- interaktiewe rekenaartegnologie soos gespesialiseerde programme wat op lessenaar-rekenaars en skootrekenaars geïnstalleer word
- applikasies ("apps") wat op slimfone en tabletrekenaars geïnstalleer word

U moet onmiddellik besef dat die dae vir ewig verby is dat u alle materiaal wat u benodig, in u handboek en by die dosent gaan kry.

U as wiskundige moet self 'n plan kan maak indien u met u Wiskunde-studie vasbrand – u en u klasmaats moet byvoorbeeld die vermoë ontwikkel om 'n soekenjin (bv. Google) in te span om inligting oor enige stuk wiskunde op te spoor. **U het nie 'n lessenaar-rekenaar of skootrekenaar hiervoor nodig nie; u selfoon het al vir jare 'n webblaaier waarmee u die**

In the old days, the study material for any typical mathematics course consisted of one or more text book and class notes, supported by additional material from libraries. The picture is, however, now very different; your study material for any modern Mathematics course will probably consist of a combination of the following:

- *Class notes*
- *One or more text books*
- *Study material obtained from trustworthy internet sources*
- *Portable calculator (with or without the ability to generate graphs and tables)*
- *Interactive computer technology such as specialized programs (software) which is installed on desktop or laptop computers*
- *Applications ("apps") which are installed on smart phones and tablet computers*

You must immediately realize that the days are forever gone when you received all the material which you require from the lecturer.

You as mathematician must be able to devise a plan whenever you get stuck with your Mathematics studies – you and your class mates must for example develop the ability to access a search engine (i.e. Google) in order to find information regarding any piece of mathematics. ***You do not need a desktop or laptop computer in order to accomplish this;***

soekenjins kan bereik en webblaaie oopmaak.

for many years your cell phone has been equipped with the ability to access a search engine and open web pages.

Webblaaie soos Wolfram Mathworld is uiters waardevol en raak elke dag meer gebruikersvriendelik.

Web pages such as Wolfram Mathworld are extremely valuable and becomes more user-friendly by the day.

U moet ook aanleer om u redenasies en berekeninge self te toets. In die meeste gevalle is die gevorderdste tegnologie wat u sal benodig 'n potlood, papier en miskien 'n sakrekenaar – Wiskunde as vakgebied is self 'n groot abstrakte stuk tegnologie.

You must also acquire the ability to privately check your reasoning and calculations. *In most cases you would need nothing more than a pencil, paper and perhaps a calculator – Mathematics itself as subject discipline functions as one enormous abstract piece of technology.*

U vermoë om onafhanklik te kan leer sal een van u magtigste gereedskapstukke in die studie van Wiskunde wees.

Your ability to study and learn independently will be one of your most powerful tools in the study of Mathematics.

0.4 'n Nuttige beskrywing van wat Wiskunde is/ *A useful description of what Mathematics is*

Wiskunde is 'n groot versameling abstrakte idees wat almal op 'n groot aantal maniere met mekaar verband hou. Hierdie verbande gaan oor die betekenis van die idees.

Mathematics is a large set of abstract ideas which are all connected to one another in a variety of ways. These connections have to do with the meaning of the ideas.

Wiskunde is onder meer 'n masjien of taal waarmee abstrakte sowel as konkrete verskynsels op 'n elegante manier beskryf kan word – 'n beskrywing so elegant dat dit nie staties is nie maar gemanipuleer kan word volgens sekere afsprake en reëls om nuwe lig op die oorspronklike verskynsel te werp. Sommige geleerdes beskryf Wiskunde as die "maak van patrone" – soms is hierdie patrone sigbaar en meetbaar en die gevolge van menslike waarnemings; ander kere is hulle onsigbaar of selfs onmeetbaar en kan hulle slegs simbolies, numeries of grafies voorgestel word deur middel van **modelle**.

Mathematics is, among other things, a machine or language with which abstract as well as concrete phenomena may be described in an elegant way – a description so elegant that it is not static but capable of being manipulated according to certain conventions and rules in order to shed new light on the original phenomenon. Some thinkers describe Mathematics as "the making of patterns" – sometimes these patterns are visible and measurable and the consequence of human observations; sometimes they are invisible and even immeasurable and can they only be represented symbolically, numerically or graphically by means of ***models****.*

Wiskundige en wetenskaplike modelle is abstrakte denkvoorstellings van werklike of denkbeeldige patrone. Hierdie modelle kan

Mathematical and scientific models *are abstract mental representations of real or* verskillende vorme aanneem, bv.

- 'n stel formules
- tabelle gevul met numeriese data
- 'n grafiese voorstelling
- 'n rekenaarsimulasie
- 'n woordelikse beskrywing

Die **Wet van Ohm** wat in Graad 9 bespreek word is 'n voorbeeld van 'n **wiskundige model** vir die gedrag van stroom, weerstand en potensiaalverskil in 'n eenvoudige elektriese stroombaan.

Hierdie wiskundige modelle werk volgens die beginsels wat in die teorie van Wiskunde geformuleer en uitgedruk word.

Wiskundige en wetenskaplike teorie is 'n baie belangrike begrip, aangesien die woord teorie in die konteks van wetenskap en Wiskunde 'n heel ander betekenis besit as die betekenis wat dit gewoonlik in die alledaagse lewe het.

In Wiskunde en wetenskap beteken die woord "teorie" NIE 'n raaiskoot of losstaande idee of vermoede nie. 'n Wiskundige of wetenskaplike teorie is 'n **robuuste stelsel samehangende idees en verbande wat suksesvol gebruik kan word om waarnemings te verklaar en voorspellings te maak**. 'n Teorie geld slegs wanneer dit sin maak en praktiese waarde het.

Drie voorbeelde van suksesvolle teorieë:

Die **teorie van eksponente en logaritmes** bevat die versameling begippe, definisies, verbande of verwantskappe, formules en betekenisse wat ons in staat stel om onder meer die eindwaarde van 'n vaste belegging teen saamgestelde rente te bereken.

imaginary patterns. These models may take different forms, for example.

- *a set of formulae*
- *tables filled with numeric data*
- *a graphical representation*
- *a computer simulation*
- *a verbal description*

*The **Law of Ohm** which was discussed in Grade 9 is an example of a **mathematical model** for the behaviour of current, resistance and potential difference in a simple electrical circuit.*

These mathematical models operate according to the principles which are formulated and expressed in the theory of Mathematics.

***Mathematical and scientific theory** is a very important concept, because the meaning of the word theory is very different in the context of Mathematics and science than its meaning in everyday life.*

***In Mathematics and science the word "theory" does NOT mean a guess or loose idea or suspicion.** A Mathematical or scientific theory is a **rigorous system of consistent ideas and relationships which may successfully be used to explain observations and make predictions**. A theory is only valid as long as it makes sense and has practical value.*

Three examples of successful theories:

*The **theory of exponents and logarithms** contains the set of concepts, definitions, connections or relationships, formulae and meanings which enable us to calculate, among other things, the final value of a fixed investment at compounding interest. Exponents and logarithms are abstract phenomena, as are the laws, formulae and*

Eksponente en logaritmes is abstrakte verskynsels, so ook die wette, formules en verbande wat daarop van toepassings is – maar die teorie werk, aangesien ons dit prakties kan gebruik om betekenisvolle antwoorde op berekeninge te kry.

Die **teorieë van gravitasie** bevat die versameling begrippe, definisies, verbande of verwantskappe, formules en betekenisse wat ons in staat stel om byvoorbeeld die gedrag van 'n vryvallende voorwerp te beskryf en selfs te voorspel. Gravitasie self is onsigbaar en so ook die universele swaartekragkonstante en die algebraïese reëls – die elemente van die Algemene Relatiwiteitsteorie is nog meer abstrak – en tog beskryf, verklaar en voorspel hierdie teorieë die waarnemings wat ons maak wanneer voorwerpe aan gravitasie onderwerp word.

Die **teorie van evolusie** bevat die versameling begrippe, definisies, wetmatighede en betekenisse wat ons in staat stel om die lang-termyn patrone wat by die studie van spesies in die natuur waargeneem word, te verklaar. Dit verklaar en beskryf die fossiele wat waargeneem word, asook die eienskappe van die DNS van organismes. Groot dele van die moderne mediese wetenskap is die gevolg van die suksesvolle toepassing van die teorie van evolusie. Hierdie teorie verklaar ook waarom die meeste moderne organismes (die mens ingesluit) se DNA tot 'n baie hoë persentasie met dié van ander organismes (selfs plante en diere) ooreenstem.

Teorie is dus uiters noodsaaklik by enige wetenskap, ook by Wiskunde: Dit verskaf die raamwerk waarbinne en die meganismes waarmee die bepaalde studieveld werk. Daarom is dit fataal as u Wiskunde sien as 'n blote versameling van resepte en reëls; Wiskunde gaan oor die patrone en verbande (konneksies) tussen abstrakte begrippe – die sogenaamde reëls en wette van Wiskunde is

relationships applicable to them – but the theory works, since we can use it practically to obtain meaningful answers to calculations.

*The **theories of gravitation** contain the set of concepts, definitions, connections or relationships, formulae and meanings which enable us to, for example, describe and even predict the behaviour of a free-falling object. Gravity itself is invisible and that is also the case with the universal constant of gravitation and the algebraic rules – the elements of the General Theory of Relativity are even more abstract – and yet, these theories describe, explain and predict the observations we make when objects are subjected to gravity.*

*The **theories of gravitation** contain the set of concepts, definitions, connections or relationships and meanings which enable us to explain the long-term patterns observed in nature in the studies of species. It explains and describes the fossils observed, as well as the properties and characteristics of the DNA of organisms. Large parts of modern medical science is the consequence of the successful application of the theory of evolution. This theory also explains why a high percentage of the DNA of most modern organisms (including humans) is similar to the DNA of other organisms (even plants and animals).*

***Theory is therefore extremely necessary in any science, including Mathematics: It provides the framework in which and the mechanisms with which the particular field of study operates.** That is why it is fatal to view Mathematics as a mere collection of recipes and rules; Mathematics deals with the patterns and relationships (connections) between abstract concepts – the so-called rules and laws of Mathematics are actually generalizations of the patterns and relationships which we discover when we work with numbers, symbols, operations, etc.*

Therefore, your knowledge and skills regarding Mathematics may not exist in

eintlik veralgemenings van die patrone en verbande wat ons ontdek wanneer ons met getalle, simbole, bewerkings ens. werk.

U kennis en vaardighede van Wiskunde mag dus nie in netjiese afsonderlike kompartemente bestaan, soos gereedskapstukke wat op die rakke van 'n pakkamer gestoor word nie.

Dit moet eerder 'n netwerk wees van begrippe en vaardighede, waar elke deel van die netwerk op baie maniere met al die ander dele van die netwerk verbind is.

So is die subvelde van Wiskunde, byvoorbeeld algebra, Euclidiese meetkunde, trigonometrie en differensiasie, kunsmatige indelings wat mense maak om die vakgebied te organiseer – **daar bestaan in werklikheid geen werklike skeidings tussen hierdie subvelde nie en elkeen van hulle is met elkeen van die ander verbind. Wanneer ons 'n wiskundige probleem oplos, gebruik ons tipies idees wat uit meer as een subveld kom**. Die bewys van 'n trigonometriese identiteit, byvoorbeeld, benodig kennis van meetkunde, trigonometrie en algebra.

U studie van Wiskunde verloop dus nie soos 'n huis wat onbeweeglike steen vir onbeweeglike steen, laag op laag, netjies van onder na bo opgebou word nie – dit verloop eerder soos 'n boom, wat van onder na bo groei deur te vertak en weer te vertak, elke takkie uniek en afsonderlik maar tog aan elke ander deel van die boom verbind as 'n vaste maar buigsame eenheid.

neat, separate compartments, like the tools stored on the shelves of a store room.

Rather, it should consist of a network of concepts and skills, where each part of the network is connected to every other part of the network in may ways.

The sub-fields of Mathematics, for example algebra, Euclidean geometry, trigonometry and differentiation, are artificial subdivisions made by humans in order to organize the subject field – ***in reality, there does not exist any partition between these sub-fields and each one of them is connected to every one of the others. When we solve a mathematical problem, we typically use ideas which we extract from more than one sub-field****. The proof of a trigonometric identity, for example, requires knowledge of geometry, trigonometry and algebra.*

Therefore, your study of Mathematics does not proceed like a house which is built by neatly stacking static bricks, layer upon layer, from bottom to top – rather, it proceeds like a tree, which grows from below to above by means of repeated branching, each branch unique and separate but yet connected to every other part of the tree as a solid but flexible unit.

0.5 Hoe om Wiskunde te bemeester/ *How to master Mathematics*

Die studie van Wiskunde verg **fokus en tyd**.

- Maak ten alle tye seker dat u elke stukkie van elke bespreking verstaan
- Wiskunde is die wetenskap van betekenis-making. **Indien u 'n stuk werk moet memoriseer omdat u die betekenis daarvan nie kan verstaan nie, dan is u nie meer met Wiskunde besig nie**.
- Indien u aanvanklik nie 'n greep op 'n begrip of metode kan kry nie, werk verder en probeer 'n voorbeeld in die hande kry waar met daardie begrip of metode gewerk word. Werk hierdie voorbeeld deur en gaan dan terug na die bespreking wat u aanvanklik nie kon verstaan nie.
- Probeer insien waar die nuwe stukkie werk inpas en met watter vorige werk wat u al gedoen het, dit verband hou. **Soek patrone en konneksies – daar is fout indien u enige nuwe stukkie werk as 'n afsonderlike aparte deeltjie beskou wat losstaan van die res van u Wiskunde-kennis**.
- **Maak gedurig planne om u antwoorde te toets**. Dit help om u begrip uit te brei.
- Raak gemaklik daarmee om saam met een of meer klasmaats te werk. Bespreek enige nuwe openbarings wat u ontvang terwyl u deur 'n stuk werk met mekaar.
- Maak gedurig u eie notas en opsommings in 'n vorm wat u self goed verstaan.
- **Bestee tyd**. Dit is die belangrikste hulpbron. Kyk weer na die aanhaling

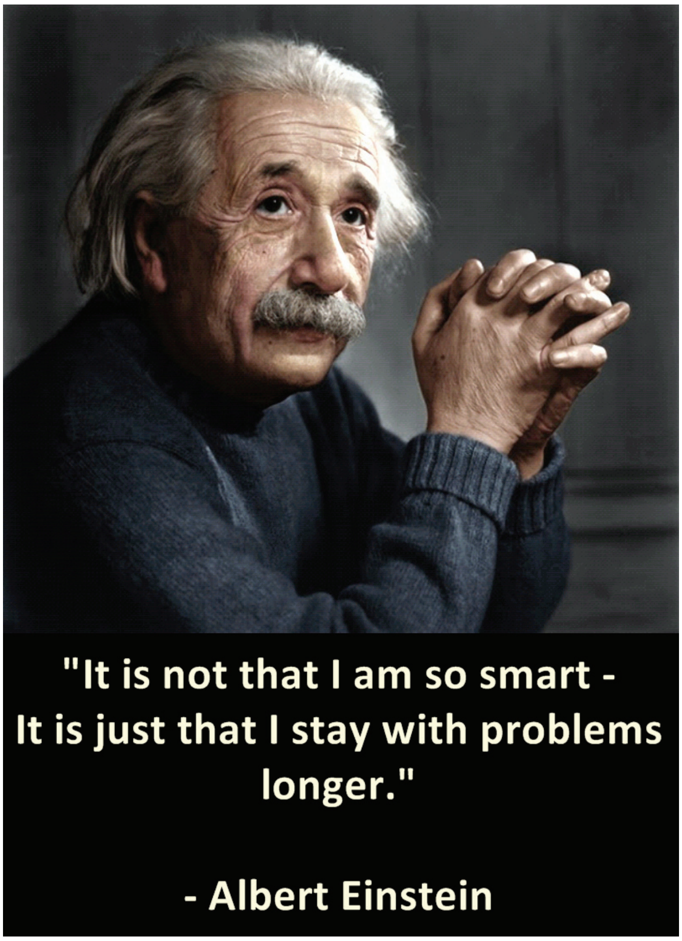
*The study of Mathematics requires **focus and time**.*

- *Be certain at all times that you understand each part of each discussion*
- *Mathematics is the science of meaning-making. **If you have to memorize a piece of work because you cannot understand its meaning, then you are no longer busy with Mathematics**.*
- *If you cannot at first establish a grip on a concept or method, work further and attempt to find an example where the particular concept or method is applied. Work through this example and then return to the discussion that you could not at first understand.*
- *Attempt to establish where the new piece of work fits in and to identify which previous work is connected to it. **Look for patterns and connections – something is wrong as soon as you consider any new piece of work as a separate isolated part which stands disconnected from the rest of your Mathematics knowledge**.*
- ***Continually devise plans by which you can check your answers**. That will help to extend and expand your understanding.*
- *You should become comfortable working with one or more of your classmates. Discuss with one another any new revelations you receive while working through a new piece of work.*
- *Continually make your own notes and summaries in a form which you*

op die voorblad van hierdie werkboek:

understand well.

- ***Invest time***. *That is your most important resource. Consider again the quote on the cover of this workbook:*



1 Algebraïese vaardighede en eksponente/ *Algebraic proficiency and exponents*

Leerdoelstellings vir hierdie leereenheid

Na afhandeling van hierdie leereenheid moet die student in staat wees om die volgende te doen:

1. **Te onderskei tussen die volgende instruksies/ aksiewoorde:**

   Vereenvoudig, faktoriseer, los op, differensieer, bewys die identiteit

2. **Te onderskei tussen die volgende konsepte:**

   Uitdrukking, term, teller, noemer, faktor, oplossing, vergelyking

3. **Die eienskappe van reële getalle en hulle bewerkings korrek toe te pas**
4. **Die absolute waarde van 'n getal te definieer en die eienskappe daarvan in berekeninge en redenasies te gebruik**
5. **Die eienskappe van eksponente te kan gebruik om uitdrukkings te vereenvoudig**
6. **Toepaslike eksponensiële vergelykings op te los deur van die eienskappe van eksponente gebruik te maak**

Learning aims for this study unit

Upon completion of this study unit the student must be able to do the following:

1. ***Distinguish between the following instructions/ action words:***

   Simplify, factorize, solve, differentiate, prove the following identity

2. ***Distinguish between the following concepts:***

   Expression, term, numerator, denominator, factor, solution, equation

3. ***Correctly apply the properties of real numbers and their operations***
4. ***Define the absolute value of a number and utilize its properties in calculations and reasoning***
5. ***Use the properties of exponents in order to simplify expressions***
6. ***Solve suitable exponential equations by exploiting the properties of exponents***

1.1 Instruksies, aksiewoorde en inleidende begrippe in wiskunde/ Instructions, action words and introductory concepts in mathematics

Wiskunde as menslike aktiwiteit is eintlik 'n sekere manier om die konkrete en abstrakte wêreld te ondersoek.

Wanneer 'n ingenieur, ekonoom, natuurwetenskaplike, onderwyser of wie ook al "wiskunde doen" beteken dit eintlik dat hy 'n situasie op 'n sekere manier beskou en volgens sekere kreatiewe beginsels probeer om daardie situasie beter te verstaan.

Dit is eintlik verskriklik kunsmatig om wiskunde te reduseer tot "die doen van somme" of selfs tot "die oplos van probleme". Tog is dit dikwels wat van ons in wiskunde-klasse of tydens wiskundige take en toetse verwag word.

Ons moet dus tog eers fokus op watter soort wiskundige probleme ons kan teëkom, sodat ons kan agterkom wat ons in elke geval moet doen om "die antwoord te kry".

In wiskunde-lesings, tydens wiskunde-oefensessies en gedurende wiskundetoetse kom ons instruksies of aksiewoorde teë wat vir ons aandui wat ons moet doen. Voorbeelde van sulke aksiewoorde is:

- **Vereenvoudig**
- **faktoriseer**
- **los op vir**
- **differensieer**
- **bewys** die identiteit

By die toepassing van wiskunde in ander vakgebiede soos rekeningkunde, ekonomie,

Mathematics as human activity is in actual fact a specific, particular way of investigating both the real world and the abstract world.

Whenever an engineer, economist, natural scientist, teacher or any other person "does mathematics" it simply means that he is approaching a situation in a certain way and that he tries to gain better understanding of that situation by proceeding according to certain creative principles.

It is really terribly artificial to reduce mathematics to "the doing of sums" or even to "the solution of problems". Still, that is often what is required of us in mathematics classes or during mathematical tasks and tests.

So, we must first focus on what kind of mathematical problems we may encounter, so that we can become aware of what we must do in each case in in order to "obtain the answer".

In mathematics lectures, during mathematics practice session and during mathematics tests we encounter instructions or action words which indicate what we must do. Examples of such action words are:

- ***Simplify***
- ***factorize***
- ***solve for***
- ***differentiate***
- ***prove** the identity*

In the application of mathematics to other fields of study such as accounting, fisika, chemie, rekenaarwetenskap en statistiek is die werklikheidsgetroue probleme wat ons moet oplos, dikwels nie in terme van aksiewoorde gedefinieer nie:

- Bepaal die eindwaarde van die belegging
- Vergelyk die vraag- en aanbod-grafieke
- Bereken die maksimumhoogte van die projektiel
- Wat die konsentrasie van die oplossing na 30 sekondes?

By die toepassing van wiskunde in ander vakgebiede moet ons dus meestal self bepaal watter wiskundige instruksies of aksiewoorde geïmpliseer word. 'n Mens leer deur ervaring hoe om werlikheidsgetroue probleme in wiskundige instruksievorm te interpreteer.

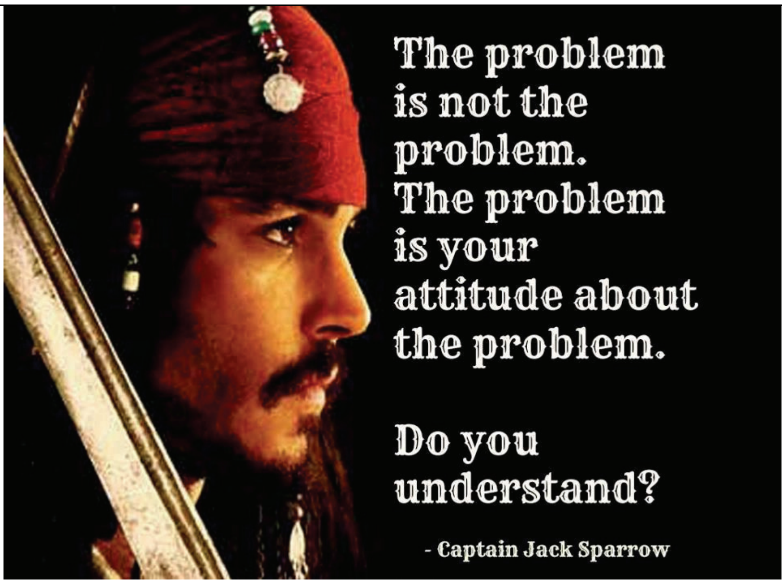
Let ten slotte daarop dat "'n probleem" in wiskunde eintlik nie iets negatiefs is wat vermy moet word, soos wat die woord in die alledaagse lewe verstaan word nie. **In wiskunde is "'n probleem" in werklikheid die instrument of gereedskapstuk** waardeur ons tot kennis en insig van wiskunde self kom, asook begrip van die situasie waaruit die probleem spruit.

economics, physics, chemistry, computer science and statistics, the real-life problems which we must solve are often not clearly defined in terms of action words:

- *Determine the final value of the investment*
- *Compare the demand- and supply-graphs*
- *Calculate the maximum altitude of the projectile*
- *What is the concentration of the solution after 30 seconds*

Usually, in the application of mathematics to other fields of study we ourselves must figure out which mathematical instructions or action words are implied. *This skill by which a person interprets real-life problems in mathematical instruction form is acquired through experience.*

Lastly, we should note that "a problem" in mathematics is actually not something negative which should be avoided – as implied by the word "problem" in everyday language. ***In mathematics "a problem" is in actual fact an instrument or tool*** *by which we acquire knowledge and insight regarding mathematics itself as well as understanding regarding the situation which gave rise to the problem.*



'n Mens leer wiskunde dus deur probleemsituasies te bestudeer en op allerlei maniere te hanteer. Meestal lê die waarde van 'n wiskundige probleem in die prosesse waardeur die probleem hanteer word, eerder as net in die oplossing. Daarom het elke wiskunde-probleem wat u doen, waarde – selfs al kry u nie die "mees korrekte antwoord" in die hande nie.

Daar is baie aksiewoorde in wiskunde waarmee u nog in u verdere studie te doen sal kry.

Maak seker dat u presies weet wat elke aksiewoord beteken en hoe die oplossing lyk wat by daardie aksiewoord (instruksie) pas.

So a person learns mathematics by studying problem situations and treating them in various ways. Most of the time, the value of a mathematical problem is contained in the processes by which the problem is handled, rather than only in its solution. That is why every mathematics problem that you do, has value – even if you are unable to obtain the "most correct answer".

There are many action words in mathematics which you will encounter during further study.

Always ensure that you know the precise meaning of each action word and also that you know the form and appearance of the solution associated with that action word (instruction).

Oefening 1.1

Beskou elkeen van die volgende gegewe probleme. By elkeen moet u self besluit wat om te doen (wat die instruksie by daardie vraag moet wees). By sommige van hulle kan daar twee of meer moontlike instruksies wees. Doen dan die berekening(s) wat by die instruksie(s) pas.

Exercise 1.1

Consider each of the following given problems. In each case, you must decide what to do (what the instruction for that question should be). In some cases there may be two or more possible instructions. Then do the calculations(s) which goes with the instruction(s).

1. $-2x^2+7x-6$

Instruksie/*Instruction*: Berekening/*Calculation*:	Instruksie/*Instruction*: Berekening/*Calculation*:

2. $(p-2)\left(p^2+2p+4\right)$

Instruksie/*Instruction*: Berekening/*Calculation*:	Instruksie/*Instruction*: Berekening/*Calculation*:

3. $\dfrac{(10-5t)(2+t)}{t-3}=0$

Instruksie/*Instruction*: Berekening/*Calculation*:	Instruksie/*Instruction*: Berekening/*Calculation*:

4. $30k^3-22k^2-28k=0$

Instruksie/*Instruction*: Berekening/*Calculation*:	Instruksie/*Instruction*: Berekening/*Calculation*:

5. $f(x) = 6\sqrt{x} - \frac{3}{x} + 5x^2$

Instruksie/*Instruction*: Berekening/*Calculation*:	Instruksie/*Instruction*: Berekening/*Calculation*:

6. $P(x) = x^3 - 6x^2 - x + 30$ en/*and* $P(3) = 0$

Instruksie/*Instruction*: Berekening/*Calculation*:	Instruksie/*Instruction*: Berekening/*Calculation*:

7. $x^3 - 6x^2 - x + 30 = 0$ en/*and* $x + 2$ is 'n faktor/*is a factor*

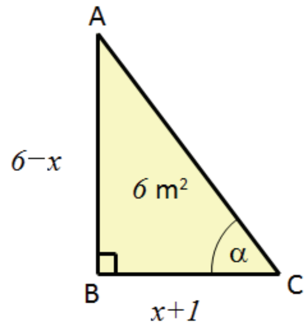
Instruksie/*Instruction*: Berekening/*Calculation*:	Instruksie/*Instruction*: Berekening/*Calculation*:

8. $\sin\theta = \sqrt{1 - \cos^2\theta}$

Instruksie/*Instruction*:

Berekening/*Calculation*:

9. Gegee/*Given*



Instruksie/*Instruction*:	Instruksie/*Instruction*:
Berekening/*Calculation*:	Berekening/*Calculation*:

Soos u kan sien, is elke probleem op sy eie manier interessant, dikwels met verskillende aspekte waarin ons kan belang stel.

As you can see, each problem is interesting in its own way, often with different aspects in which we might be interested.

1.2 Die reële getalle/ *The real numbers*

Getallestelsels

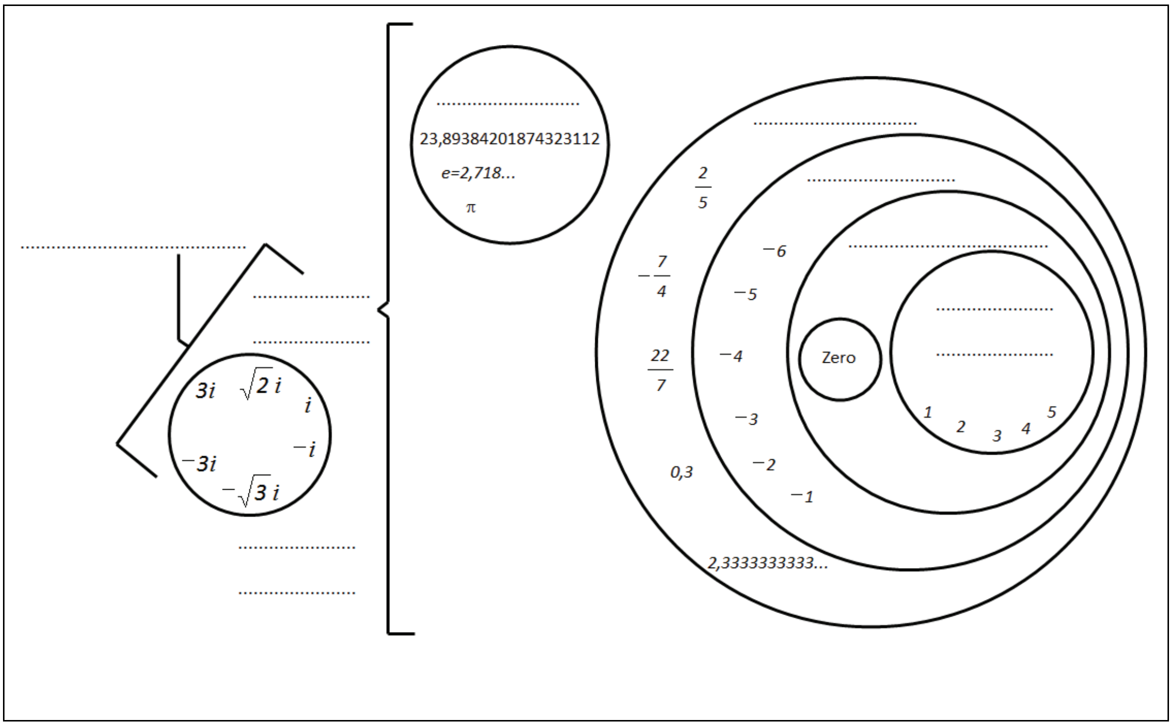
Ons klassifiseer getalle (abstrakte mensgemaakte simboliese en konseptuale voorstellings wat aantal en hoeveelheid voorstel) volgens hulle eienskappe soos volg:

(Voorsien die beskrywende name van elkeen van die volgende versamelings)

Number systems

We classify numbers (abstract man-made symbolic and conceptual representations which indicate amount and quantity) as follows according to their properties:

(Supply descriptive names for each of the following sets)



Begenoemde verwantskappe word ook in versamelingsnotasie soos volg geskryf:

The relationships above may also be expressed in set notation as follows:

$\mathbb{N}=\{1;2;3;4;...\}$	(natuurlike getalle/ *natural numbers*)
$\mathbb{N}_0=\{0;1;2;3;4;...\}$	(telgetalle/ ***whole** numbers*)
$\mathbb{Z}=\{...;-4;-3;-2;-1;0;1;2;3;4;...\}$	(**heel**getalle/ *integers*)
$\mathbb{Q}=\left\{x \middle\vert x=\frac{p}{q},\ p\in\mathbb{Z}, q\in\mathbb{Z}, q\neq 0\right\}$	(rasionale getalle/ *rational numbers*)

$\mathbb{I}=\{x \mid x\in\mathbb{R}, x\notin\mathbb{Q}\}$	(irrasion**a**le getalle/ *irrational numbers*)
$\mathbb{R}=\{x \mid x\in\mathbb{Q}$ of / or $x\in\mathbb{I}\}$	(reële getalle/ *real numbers*)

Dit volg dat al die versamelings hierbo genoem deelversamelings is van die reële getalle, d.w.s. $\mathbb{N}\subset\mathbb{N}_0\subset\mathbb{Z}\subset\mathbb{Q}\subset\mathbb{R}$ en $\mathbb{I}\subset\mathbb{R}$

Verder volg dat $\mathbb{Q}\cup\mathbb{I}=\mathbb{R}$ en $\mathbb{Q}\cap\mathbb{I}=\varnothing$.

Wat beteken bogenoemde in woorde? U **MOET** weet!

It follows that all the sets above are subsets of the set of real numbers, in other words: $\mathbb{N}\subset\mathbb{N}_0\subset\mathbb{Z}\subset\mathbb{Q}\subset\mathbb{R}$ *and* $\mathbb{I}\subset\mathbb{R}$

Further, it also follows that $\mathbb{Q}\cup\mathbb{I}=\mathbb{R}$ *and* $\mathbb{Q}\cap\mathbb{I}=\varnothing$.

*What does the symbology above mean in words? You **MUST** know!*

Basiese eienskappe van die reële getalle

Gestel a en b en c is reële getalle. Dan geld:

1. **Eienskappe van die getal nul (optellingsidentiteitselement)**

   1.1 $a+0=$...............

   1.2 $a\times 0=$................

   1.3 $\frac{0}{a}=$....... met $a\neq 0$

   1.4 $\frac{a}{\ldots\ldots\ldots}$ is ongedefineerd

   1.5 As $ab=$......... dan is $a=0$ of $b=0$

   (.........-produk-stelling)

   1.6 As $\frac{a}{b}=0$ dan is $a=$.......... en $b\neq$............

2. **Bewerkingseienskappe van reële getalle**

   2.1 Geslotenheid

   $a+b\in\mathbb{R}$ en $ab\in\mathbb{R}$

   2.2 Kommutatiwiteit

   $a+b=$............. en $ab=$...........

Basic properties of the real numbers

Suppose a *and* b *and* c *are real numbers. Then the following hold:*

1. ***Properties of the number zero (additive identity element)***

   1.1 $a+0=$...............

   1.2 $a\times 0=$................

   1.3 $\frac{0}{a}=$........ *with* $a\neq 0$

   1.4 $\frac{a}{\ldots\ldots\ldots}$ *is undefined*

   1.5 *If* $ab=$......... *then* $a=0$ *or* $b=0$

   (...........-product theorem)

   1.6 *If* $\frac{a}{b}=0$ *then* $a=$......... *and* $b\neq$............

2. ***Operational properties of real numbers***

   2.1 *Closure*

   $a+b\in\mathbb{R}$ *and* $ab\in\mathbb{R}$

   2.2 *Commutativity*

   $a+b=$............. *and* $ab=$...........

2.3 Assosiatiwiteit

$\left(a+b\right)+c=a+\left(b+c\right)$ en
$\left(ab\right)c=a\left(bc\right)$

2.4 Identiteit:

$a+0=$.......... en
$a\times 1=$...........

2.5 Inverse:

$a+\left(-a\right)=$....... en
$a\times\frac{1}{a}=$........
waar $a\neq$........ by $a\times\frac{1}{a}$

2.6 Distributiwiteit:

$a\left(b\pm c\right)=$..........................

3. Volgorde van algebraïese bewerking (berus op afsprake en patrone)

3.1 Vir herhaalde optelling en aftrekking werk ons van links na regs.

3.2 Vir herhaalde vermenigvuldiging en deling werk ons van links na regs.

3.3 Vir gekombineerde bewerkings kan ons nie net van links en regs werk nie, maar word optelling en aftrekking laaste gedoen. Uitdrukkings in hakies word eerstens verreken – dan vermenigvuldiging en deling (gelyke prioriteit) en laastens word opgetel en afgetrek.

3.4 Vir gekombineerde

2.3 Associativity

$\left(a+b\right)+c=a+\left(b+c\right)$ *and*
$\left(ab\right)c=a\left(bc\right)$

2.4 Identity:

$a+0=$.......... *and*
$a\times 1=$...........

2.5 Inverse:

$a+\left(-a\right)=$....... *and*
$a\times\frac{1}{a}=$........
where $a\neq$........ *in* $a\times\frac{1}{a}$

2.6 Distributivity:

$a\left(b\pm c\right)=$..........................

3. Order/ priority of algebraic operations (based on conventions and patterns)

3.1 For repetitive addition and subtraction we work from left to right.

3.2 For repetitive multiplication and division we work from left to right.

3.3 For combined operations we cannot simply proceed from left to right, but we should take care to perform addition and subtraction last. Expressions in brackets must be evaluated first – then multiplication and division and lastly addition and subtraction.

3.4 For combined operations

bewerkings is die volgorde soos volg:

the order is as follows:

Prioriteit/ ***Priority***	**Bewerking/** ***Operation***	**Verduideliking** ***Explanation***
1	() hakies/ *parentheses*	**Met hakies binne hakies word die binneste hakies eerste verwyder/** ***With sets of parentheses inside other sets of parentheses, proceed from inside out***
2	Magsverheffing en Worteltrekking/ *Raising to powers and applying radical operations*	Moet as spesiale hakies beskou word/ *Should be treated as a special case of parentheses*
3	Van/ *of*	vervang met ‘n $\times$-teken/ *substitute with a multiply-sign*
4	$\times$ en/ *and* /of/ *or* $\div$	Gelyke prioriteit/ *same priority*
5	$+$ en/ *and* /of/ *or* $-$	Gelyke prioriteit/ *same priority*

Sommige wiskundiges vereenvoudig bogenoemde skema soos volg:

Some mathematicians simplify the scheme above as follows:

Identifiseer alle plus en minus tekens buite hakies. Evalueer dan alle uitdrukkings tussen plus en minus tekens (faktore) en tel laastens op en trek af.

Identify all plus and minus signs outside brackets. Then evaluate all expressions between plus and minus signs and lastly, add and subtract.

Die belangrikste sake wat u omtrent algebra moet verstaan, is dat:

- enige uitdrukking of formule uit terme saamgestel is – die terme word deur plustekens en minustekens geskei (wortels,

The most important matters which you must grasp regarding algebra is:

- *Any expression or formula consists of terms – the terms are separated by plus or minus signs (radicals, exponents, multiplication and* eksponente, maal en deel skei NIE terme van mekaar nie)

- Wat dus ook al tussen plustekens en minustekens staan, moet as een getal gelees en verstaan word. 'n "Algebraïese terme" beteken "stukke van 'n uitdrukking wat deur plus en minus van mekaar geskei word".
- Hakies groepeer alles wat daarbinne saam as een getalwaarde – daarom bereken ons altyd die inhoud van 'n stel hakies heel eerste.

'n horisontale deelteken (lyntjie met teller bo en noemer onder) plaas outomaties alles in die teller en alles in die noemer in hakies.

division does NOT separate terms)

- *Whatever stand between plus or minus signs must be regarded as one number. "Algebraic terms" mean "pieces of an expression which is separated by plus or minus signs".*
- *Parentheses group everything within together as one numeric value – that is why we always evaluate the contents of a set of parentheses first.*
- *A horizontal division sign automatically places everything in the numerator and in the denominator in brackets.*

4. Ontoelaatbare bewerkings by reële getalle

4. Illegal or non-permissible operations and real numbers

Vereenvoudig die volgende uitdrukking:

Simplify the following expression:

$$\frac{3}{7+\sqrt[3]{-343}}+\log_{10}(-10)+\frac{7}{\log_{10}(1)}-2\sqrt{3+\sqrt[3]{-64}}+\sin^{-1}(2)+\cos^{-1}(-1{,}5)-\tan(90^\circ)$$

$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Gebruik u bevindinge en maak 'n lys van ontoelaatbare bewerkings:

....

....

Use your findings and make list of non-permissible operations:

....

....

....

....

....

....

Ons moet altyd bedag wees op gevalle soos hierbo, waar getalle interessante ongewone gedrag vertoon.

We must always stay aware of cases like above, where numbers exhibit interesting unusual behaviour.

Oefening 1.2

Exercise 1.2

1. Bepaal die waarde van die volgende uitdrukkings (met ander woorde, **evalueer** die volgende) sonder enige sakrekenaar:

1. Determine the value of the following expressions (in other words, ***evaluate*** *the following) without a calculator:*

1.1 $\frac{3(4)}{2}+6(2+3)-\frac{27-7}{7+3}$

=...

=...

1.2 $4-3|(-2)(3)|+\frac{5^3}{25}-\frac{12}{2+\frac{2}{5}}-\sqrt{5^2-3^2}$

Die simbool $|'n\ reële\ getal|$ staan as "die absolute waarde-bewerking" bekend - in die algemeen is $|'n\ reële\ getal| = positiewe\ waarde\ van\ daardie\ reële\ getal$. Later meer hieroor./

The symbol $|a\ real\ number|$ *is known as "the absolute value operation" - in general, it holds that* $|a\ real\ number| = positive\ value\ of\ that\ real\ number$. *Later more about this.*/

=..

=..

=..

1.3 $$\frac{2\sqrt{169-144}}{5}+\frac{\left|(8-16)\right|}{2^3}-\sqrt{\left|64-128\right|}+\frac{100-25}{(10-5)^2}$$

=..

=..

=..

1.4

$$\frac{\left(\frac{7}{\sqrt{144}}-\frac{6^0}{\sqrt[3]{216}}\right)}{\left(\frac{\sqrt[3]{1000}}{\left(48-3(2)^3\right)}\right)}+\left(\frac{1}{\frac{1}{2}-\frac{\frac{1}{2}}{2}+\frac{5^0}{\sqrt{64}-\frac{2^3}{5-\frac{12-3}{9-\sqrt{\frac{72}{\frac{5}{2}-\frac{1}{2}}}}}}}\right)^{\sqrt{12-8}}-3\left(12-\frac{5}{\left(\frac{1}{2}\right)}\right)^5+\left(\frac{10^2}{103}\right)^2\cdot\left(1+\frac{3}{100}\right)^2$$

=..

=..

....

....

1. Bepaal die waarde van die volgende uitdrukkings (met ander woorde, **evalueer** die volgende) sonder enige sakrekenaar:

2. Determine the value of the following expressions (in other words, ***evaluate*** *the following) without a calculator:*

2.1 $\dfrac{4-\sqrt{16}}{3+2}$ = ...

2.2 $\dfrac{12-8}{10-\sqrt{100}}$ = ...

2.3 $5-\sqrt{16-25}$ = ...

2.4 $3-2\sqrt{169-25}$ = ...

1.3 Eksponente/ *Exponents*

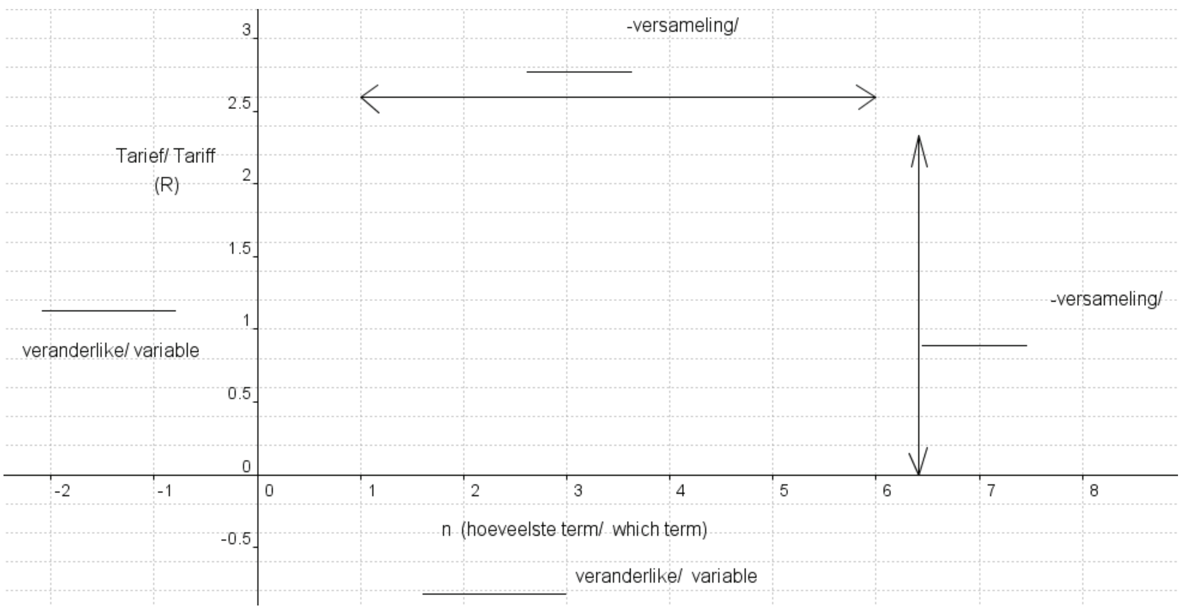
'n Maatskappy beoog om vir die volgende vyf jaar 'n tariefverhoging van 20% per jaar in te stel op 'n diens wat in 2015 per eenheid R 1,10 gekos het.

Voltooi die volgende tabel en grafiek:

A company intends to apply a tariff increase of 20% per year for the next five years on a service which cost R 1,10 *in 2015.*

Complete the following table and graph:

Jaartal/*Year*	2015	2016	2017	2018	2019	2020
Term nr. (n)	1	2	3	4	5	6
Tarief/ *Tariff*	1,10					



Lê die punte in 'n reguit lyn?

....

Bogenoemde is 'n voorbeeld van **eksponensiële gedrag**; dit het met herhaalde vermenigvuldiging van 'n sekere konstante waarde te doen.

Kan u 'n formule vir vergelyking van u grafiek in terme van n neerskryf?

....

Die formule vir die algemene term van 'n meetkundige ry is 'n voorbeeld van 'n **eksponensiële funksie**.

Ons definieer in die algemeen

Do the points lie in a straight line?

....

*The situation above is an example of **exponential behaviour**; it deals with repetive multiplication by a certain constant value.*

Are you able to write down an equation for the graph in terms of n ?

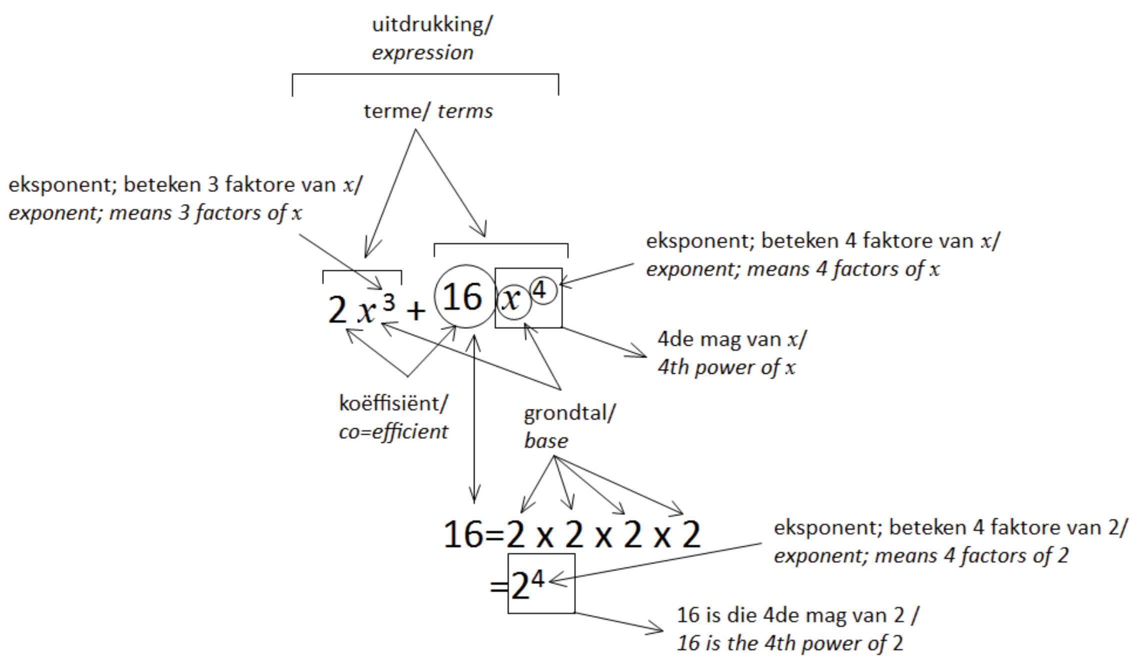
....

*The formula for the general term of a geometric sequence is an example of an **exponential function**.*

We define in general

$$\underbrace{x \times x \times x \times x \times \ldots \times x}_{n \text{ faktore van } x /\ n \text{ factors of } x} = x^n$$

As bv / *if for example* $2x^3 + (2x)^4 = 2x^3 + 16x^4$ dan/ *then*



Gebruik u kennis en voltooi die ontbrekende inligting in die volgende tabel:

Use your knowledge and complete the omitted information in the following table:

Eienskappe van eksponente ***Properties of exponents***

Eienskap/ *Property:*	**Voorbeeld/ *Example***	**Pasop/ *Beware***
$a^0 =$..............	$10^0 =$..............	$0^0 =$...........................
$a^1 =$..............	$5^1 =$..............	$0^1 =$...........................
$a^m \times a^n =$..............	$10^2 \times 10^3 = 100\,000 = 10^{\cdots\cdots}$	$a^m \times a^n \neq a^{mn}$
$\frac{a^m}{a^n} =$..............	$\frac{10^6}{10^3} = \frac{1000000}{1000} = 10^{\cdots\cdots}$	$\frac{a^m}{a^n} \neq a^{\frac{m}{n}}$
$\left(\frac{a^m}{b^n}\right)^p =$....................	$\left(\frac{10^3}{2^2}\right)^4$ $=\left(\frac{1000}{4}\right)^4$ $=250^4 =$....................... $\frac{10^{12}}{2^8}$ $=\frac{2^{12} \times 5^{12}}{2^8} = 2^4 \times 5^{12}$ $=$..................................	$\left(\frac{a^m}{b^n}\right)^p \neq \left(\frac{a}{b}\right)^{mp-np}$
$(a)^{-n} =$....................	$2^{-3} =$.................... $\left(\frac{3}{2}\right)^{-4} =$....................	$(a)^{-n} \neq -a^n$ $(a)^{-n} \neq a^{\frac{1}{n}}$
$\sqrt[n]{a^m} =$........................	$\sqrt{x} =$........................ $\sqrt[3]{y^6} =$.......................	$\sqrt[n]{a^m} \neq a^{\frac{n}{m}}$ $\sqrt[n]{a^m} \neq a^{m-n}$
$\sqrt[n]{ab} =$.........$\times$..........	$\sqrt{64t^8} = \sqrt{\ldots\ldots} \times \sqrt{\ldots\ldots}$ $=$	$\sqrt[n]{a+b} \neq \sqrt[n]{a} + \sqrt[n]{b}$ $\sqrt{x^2 \pm y^2} =$..........................

Bogenoemde mag moontlik eenvoudig en onskouspelagtig lyk, maar die foute wat met die toepassing van hierdie beginsels gemaak word, **gaan die verstand te bowe**.

*The principles above may seem simple and unspectacular, but the errors committed in their application **challenge the imagination**.*

Oefening 1.3

Exercise 1.3

1. Vereenvoudig sonder 'n sakrekenaar:

1. Simplify without using a calculator:

1.1 $4(a+b)^4 - 4(a \times b)^2$

=

1.2 $\left(\frac{3}{2}x^2\right)\left(\frac{3}{2}x^2\right)\left(\frac{3}{2}x^2\right)-\left(\frac{3}{2}x^2\right)^3-\left(\frac{3}{2}x^2+\frac{3}{2}x^2+\frac{3}{2}x^2\right)^3$

=

1.3 $\left[\left(z^2\right)^2\right]^{-1}\cdot\left(z^{-1}\right)^3$

=

1.4 $(p-m)\left(p^2+pm+m^2\right)$

=...

=...

=...

=...

$=(.........)^3-(..........)^3$

1.5 $(2x+3y)(4x^2-6xy+9y^2)$

=...

=...

=...

=...

$=(.........)^3+(..........)^3$

1.6 $(a-b)^3$

=...

=...

=...

=...

1.7 $\dfrac{t^3+t^2}{t^5}$

=

1.8 $\dfrac{r^3+r^3}{2r^2+3r^2}$

1.9 $\dfrac{3^{\alpha}\cdot 9^{\alpha+1}}{27^{\alpha+2}}$

1.10 $\dfrac{5^{\beta+2}+5^{\beta+1}}{5^{\beta+2}-5^{\beta+1}}=\dfrac{\ldots\ldots\cdot\ldots\ldots+\ldots\ldots\cdot\ldots\ldots}{\ldots\ldots\cdot\ldots\ldots-\ldots\ldots\cdot\ldots\ldots}=\dfrac{5^{\beta}(\ldots\ldots\ldots+\ldots\ldots\ldots)}{5^{\beta}(\ldots\ldots\ldots-\ldots\ldots\ldots)}$

$=\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$=\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$=\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

---

2. In Matriek het u die volgende differensiasiereël teëgekom:

2. In Matric you encountered the differentiation rule:

$$\frac{dy}{dx}\left(ax^{n}\right)=an\cdot x^{n-1}$$

Bv.

For example

$$\frac{dy}{dx}\left(3x^{4}-5x^{2}+\frac{1}{2}x-9\right)=12x^{3}-10x+\frac{1}{2}$$

Bereken nou $g'(t)$ indien...

Now calculate $g'(t)$ if...

2.1 $g(t)=\sqrt{t}+\dfrac{1}{t}$

2.2 $g(t)=4\sqrt[3]{t^2}-\dfrac{5}{3t^2}+\dfrac{2}{3\sqrt{t}}+\pi$

3. Vereenvoudig: *3. Simplify:*

3.1 $\dfrac{4^x \times 5^x}{20}+\dfrac{10^{3y}}{\left(2^y \times 5^y\right)^2}$

1.4 Eenvoudige eksponensiële vergelykings/ *Simple exponential equations*

Uit die eienskappe van eksponente kan ons sekere vergelykings oplos, waar eksponente betrokke is.

By exploiting the properties of exponents we may easily solve certain equations where exponents are involved.

Oefening 1.4

1. Bereken sonder 'n sakrekenaar die waarde van die onbekende:

Exercise 1.4

1. Calculate the value of the unknown variable without the use of a calculator:

1.1 $9^{y-2} = 27^{1-2y}$

1.2 $4^{3p} - 32^{2p-5} = 0$

1.3 $3 \cdot 10^{x} - 0{,}03 = 0$

1.4 $6 \cdot 2^{2x} - 2^{x} - 1 = 0$

Verklaar waarom daar slegs een oplossing is/ *Explain why there is only one solution*.

1.5 $2r^5 = -\frac{243}{16}$

Hoe verskil hierdie vergelyking van die vorige vier?/ *How is this equation different from the previous four?*

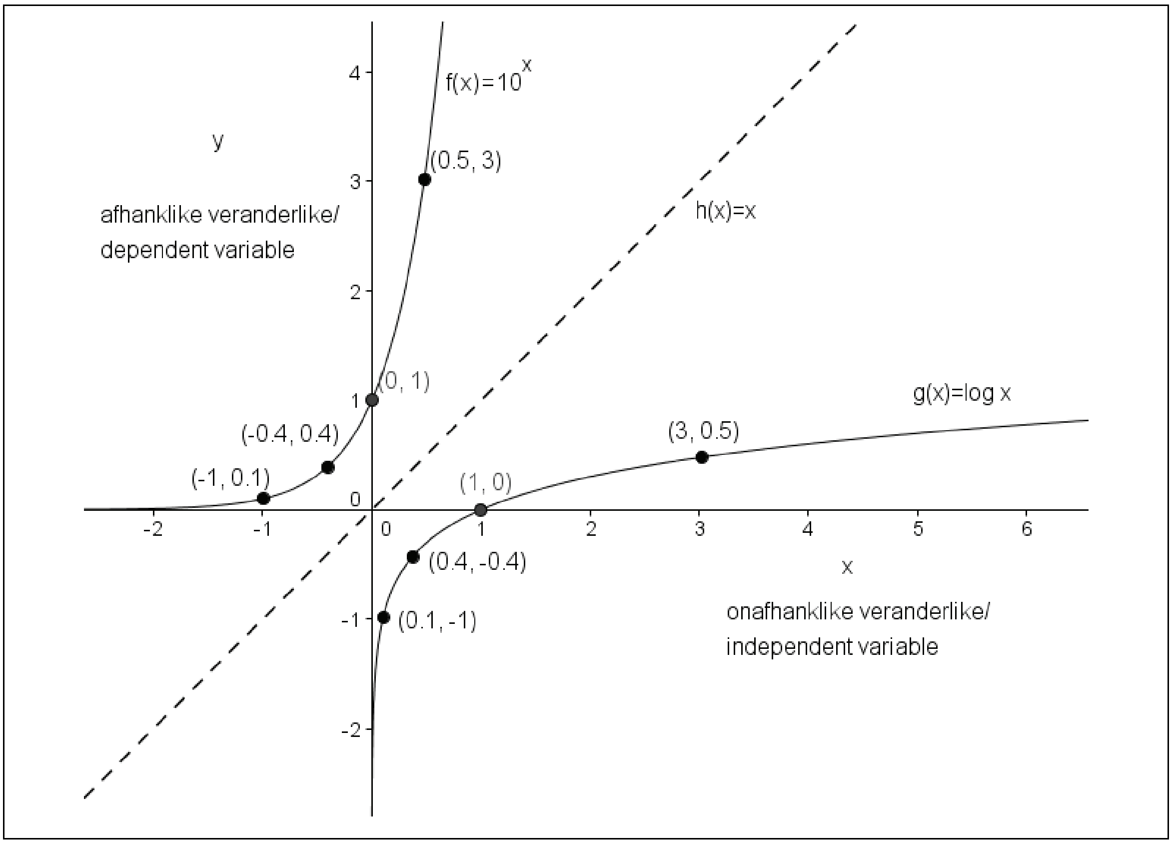
2 Logaritmes/ *Logarithms*

Leerdoelstellings vir hierdie leereenheid	***Learning aims for this study unit***
Na afhandeling van hierdie leereenheid moet die student in staat wees om die volgende te doen:	***Upon completion of this study unit the student must be able to do the following:***
1. Die eienskappe van logaritmes te kan gebruik om uitdrukkings te vereenvoudig	***1. Use the properties of logarithms in order to simplify expressions***
2. Die eienskappe van logaritmes te kan gebruik om logaritmiese vergelykings op te los	***2. Use the properties of logarithms in order to solve logarithmic equations***
3. Ingewikkelder eksponensiële vergelykings op te los deur van die verbande tussen eksponente en logaritmes gebruik te maak	***3. Solve more complicated exponential equations by exploiting the relationships between exponents and logarithms***

2.1 Logaritmes en eksponente/ *Logarithms and exponents*

Beskou onderstaande voorstelling:

Consider the represention below:



Watter waarnemings en gevolgtrekkings kan gemaak word i.v.m. die eksponensiële bewerking en die logaritmiese bewerking?

Which observations and conclusions may be made with respect to the exponential operation and the logarithmic operation?

Logaritmes is basies ‘n ander manier om dieselfde inligting te verstrek as wat deur ‘n eksponensiële vergelyking gegee word.

Logarithms basically represent another way to express the same information as that which is given by an exponential equation.

Om ‘n logaritme te definieer maak ons gebruik van die teorie van inverse funksies:

In order to define a logarithm we employ the theory of inverse functions:

Beskou die grafiek hierbo.

Consider the graph above.

As $f(x)=10^x$ dan kan ons dit skryf $y=10^x$. Om die inverse funksie van $f(x)=10^x$ te verkry, ruil ons die afhanklike en die onafhanklike veranderlike:

$x=10^y$

If $f(x)=10^x$ *then we may write that* $y=10^x$ *. In order to obtain the inverse function of* $f(x)=10^x$ *we switch dependent and independent variables:*

$x=10^y$

Maak nou vir y die onderwerp deur te definieer dat $y = \log_{10} x$

Now make y the subject by defining that $y = \log_{10} x$

Dit lewer die grafiek g hierbo, wat die spieëlbeeld is van f in die lyn $h(x) = x$.

This yields the graph g above, which is the mirror image of f in the line $h(x) = x$.

Ons definieer in die algemeen:

As $p = a^x$ met $a > 0$ en $a \neq 1$ dan is $p > 0$ vir $x \in \mathbb{R}$ en $x = \log_a p$

Voorbeelde:

$$8 = 2^3 \Leftrightarrow 3 = \log_2 8$$

$$1000 = 10^3 \Leftrightarrow 3 = \log 1000$$

$$0{,}008 = \frac{1}{125} = 5^{-3} \Leftrightarrow -3 = \log_5 (0{,}008)$$

We define in general:

If $p = a^x$ with $a > 0$ and $a \neq 1$ then $p > 0$ for $x \in \mathbb{R}$ and $x = \log_a p$

Examples:

$$8 = 2^3 \Leftrightarrow 3 = \log_2 8$$

$$1000 = 10^3 \Leftrightarrow 3 = \log 1000$$

$$0{,}008 = \frac{1}{125} = 5^{-3} \Leftrightarrow -3 = \log_5 (0{,}008)$$

Eienskappe van logaritmes/		***Properties of logarithms***
Eienskap/ *Property:*	**Voorbeeld/ *Example***	**Verklaring/ *Explanation***
$\log_a 1 = \ldots\ldots\ldots\ldots\ldots$	$\log_3 1 = \ldots\ldots\ldots\ldots\ldots$	"$\log_3 1$" beteken/ *means:* $3^{\text{WAT}/\mathit{WHAT}} = 1$?
$\log_a a = \ldots\ldots\ldots\ldots\ldots$	$\log_e e = \ldots\ldots\ldots\ldots\ldots$	"$\log_7 7$" beteken/ *means:* $7^{\text{WAT}/\mathit{WHAT}} = 7$?
$\log_a \left(\frac{1}{a}\right) = \ldots\ldots\ldots\ldots\ldots$	$\log_5 \left(\frac{1}{25}\right) = \ldots\ldots\ldots\ldots\ldots$	"$\log_7 \frac{1}{343}$" beteken/ *means:* $7^{\text{WAT}/\mathit{WHAT}} = \frac{1}{343} = \left(\frac{1}{7}\right)^3$ $= 7^{-3}$?

$\log_a(xy) = ...$	$\log 8 + \log 125$ $= \log ....$ $= ....$	$a^x \times a^y = a^{..........}$
$\log_a\left(\frac{x}{y}\right) = ...$	$\log_5 500 - \log_5 20$ $= \log_5 ....$ $= ....$	$\frac{a^x}{a^y} = a^{..........}$
$\log_a\left(x^m\right) = ...$	$\log_5 625$ $= \log_5 5^4$ $= .... \times ....$ $= ....$	$\left(a^m\right)^x = a^{..........}$
$\log_a b = \frac{\log_c b}{\log_c a}$ Vir enige/ *For any* $c > 0$	$\log_{32} 64$ $= \frac{\log_{.....} 64}{\log_{.....} 32}$ $= \frac{\quad}{\quad}$ $= ....$	$64 = 2^6$ $\therefore \log_2 64 = 6$ $32 = 2^5$ $\therefore \log_2 32 = 5$ $32^{\frac{6}{5}} = ...$
$\log_a b = \frac{1}{\log_b a}$	$\log_3 27 = \frac{1}{\log_{27} 3}$	$\log_3 27 = \frac{1}{\log_{27} 3}$ $= \frac{1}{\left(\frac{\log_{27} 3}{\log_{27} 27}\right)}$ $= \frac{\log_{27} 27}{\log_{27} 3}$
$a^{\log_a x} = ...$	$10^{\log_{10} p} = ...$	Die logaritme-bewerking is die .... van die .... bewerking/ *The logarithmic operation is the* *................ of the* *............................. operation*

Eienskappe wat NIE geld by logaritmes nie/*Properties which does NOT hold for logarithms*		
Eienskap/ *Property:*	**Voorbeeld/ *Example***	**Korrekte vorm/ *Correct form***
$\log_a (x \pm y) \neq \log_a x \pm \log_b y$	$\log_{10}(1000 \pm 100)$ $\neq \log_{10} 1000 \pm \log_{10} 100$	$\log_a (xy) = \log_a x + \log_a y$
$\log_a (xy) \neq \log_a x \times \log_b y$	$\log_{10}(1000) \neq \log_{10} 100 \times \log_{10} 10$	$\log_a (xy) = \log_a x + \log_a y$
$\log_a \frac{x}{y} \neq \frac{\log_a x}{\log_a y}$	$\log_3 \frac{81}{27} \neq \frac{\log_3 81}{\log_3 27}$	$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$
$(\log_a x)^m \neq m \cdot \log_a x$	$(\log_5 25)^3 \neq 3 \cdot \log_5 25$	$\log_a (x^m) = m \cdot \log_a x$ of / *or* $\log_a x^m = m \cdot \log_a x$

Dit is belangrik om die implikasies van bogenoemde baie goed te begryp, aangesien ons dit by baie toepassings benodig.

It is important to grasp very well the implications of the discussion above, since we require it in many applications.

Oefening 2.1

Exercise 2.1

1. Vereenvoudig sonder 'n sakrekenaar:

1. Simplify without using a calculator:

1.1 $\log_{10} 20 + \log_{10} 50$

1.2 $\log_6 36 + \log_6 \frac{1}{36} - 3\log_6 1$

1.3 $\frac{\log 81 - \log 16}{\log 9 - \log 4}$

1.4 $\log_4 4^{20} + \left(\log_4 4\right)^{20} - \left(\log_4 16\right)\left(\log_4 64\right)$

1.5 $\left(\log_7 49\right)\cdot\log_7 1$

1.6 $\left(\log 100\right)\cdot\log\left(1000\right)$

1.7 $3^{\log_3 x}$

Die natuurlike grondtal e

Blaai na die bylae van hierdie boek en voltooi Werkkaart 1.

2. Vereenvoudig sonder 'n sakrekenaar en gebruik slegs in die laaste stap 'n sakrekenaar:

The natural base e

Turn to the addendum of this book and complete Worksheet 1.

2. Simplify without a calculator but use only in the final step a calculator:

2.1 $e^2+3e^2+e^2\cdot e^{-2}-\dfrac{e^5}{e^3}+e^0$

2.2 $\frac{\ln x + 2\ln x}{\ln x^3} - \ln\left(e^3\right) + \left(\ln e\right)^3 + \ln 1$

3. Bepaal die waarde van die onbekende (Los op):

3. Determine the value of the unknown (Solve):

3.1 $\log_3 x = 4$

3.2 $\log x + \log\left(x+3\right) = 1$

3.3 $\log(1-2x)-\log(x+2)=\log 1$

3.4 $\ln x=\ln(2x-1)+2\ln x$

2.2 Die oplos van ingewikkelder eksponensiële vergelykings d.m.v logaritmes/ *Solving more complicated exponential equations using logarithms*

Aangesien die definisie van 'n logaritme in terme van eksponente gedoen is, kan ons ingewikkelde eksponensiële vergelykings oorskryf na logaritmiese vorm. Sulke logaritmiese vergelykings is dan makliker om op te los as wat die oorspronklike eksponensiële vergelyking was.

Because we defined a logarithm in terms of exponents, we may rewrite more complicated exponential equations to logarithmic form. Such logarithmic equations are then easier to solve than the original exponential equation.

Voorbeeld/ *Example*

1. Los op vir x / Solve for x : $\left(\frac{3}{10}\right)^{5x} = \frac{343}{64}$

2. Los op vir t / Solve for t : $3{,}5 \cdot 0{,}1^{3t} = 2{,}5^{t-1}$

Oplossing/ *Solution*

1.
$$\left(\frac{3}{10}\right)^{5x} = \frac{343}{64}$$
$$\therefore 5x = \log_{\frac{3}{10}} \frac{343}{64}$$
$$\therefore 5x = \frac{\log\left(\frac{343}{64}\right)}{\log\left(\frac{3}{10}\right)}$$
$$\therefore x = \frac{\left[\dfrac{\log\left(\frac{343}{64}\right)}{\log\left(\frac{3}{10}\right)}\right]}{5}$$
$$= -0{,}279$$

2. $$3{,}5\cdot 0{,}1^{3t}=2{,}5^{t-1}$$
$$\therefore \log\left(3{,}5\cdot 0{,}1^{3t}\right)=\log\left(2{,}5^{t-1}\right)$$
$$\therefore \log(3{,}5)+\log\left(0{,}1^{3t}\right)=\log\left(2{,}5^{t-1}\right)$$
$$\therefore \log(3{,}5)+3t\cdot\log(0{,}1)=(t-1)\cdot\log(2{,}5)$$
$$\therefore \log(3{,}5)+3t\cdot\log(0{,}1)=t\cdot\log(2{,}5)-\log(2{,}5)$$
$$\therefore 3t\cdot\log(0{,}1)-t\cdot\log(2{,}5)=-\log(2{,}5)-\log(3{,}5)$$
$$\therefore t\left[3\log(0{,}1)-\log(2{,}5)\right]=-\log(2{,}5)-\log(3{,}5)$$
$$\therefore t=\frac{-\log(2{,}5)-\log(3{,}5)}{3\log(0{,}1)-\log(2{,}5)}$$
$$=0{,}277$$

Oefening 2.2 | ***Exercise 2.2***

1. Los die volgende vergelykings op: | *1. Solve the following equations:*

1.1 $0{,}002\,93=3\cdot\left(\frac{1}{2}\right)^{t}$

1.2 $\frac{1}{3}\cdot\left(\frac{4}{3}\right)^{n-1}=2{,}497\,180$

1.3 $4\cdot 2^{3t} = \frac{1}{2}\cdot 5^{2t-1}$

1.4 $2\cdot\left(\frac{1}{3}\right)^{2n} - \frac{1}{2}\cdot 5^{1-3n} = 0$

Blaai na die bylae van hierdie boek en voltooi Werkkaart 2.

Turn to the addendum of this book and complete Worksheet 2.

3 Inleiding tot funksies/ *Introduction to functions*

Leerdoelstellings vir hierdie leereenheid

Na afhandeling van hierdie leereenheid moet die student in staat wees om die volgende te doen:

1. **Die formele definisie van 'n funksie as 'n spesiale relasie kan toepas**
2. **Die definisie- en waardeversameling van 'n funksie kan identifiseer**
3. **Die inverse van 'n gegewe funksie kan bepaal**
4. **Bewerkings met funksies kan uitvoer**

Learning aims for this study unit

Upon completion of this study unit the student must be able to do the following:

1. ***Apply the formal definition of a function as a special relation***
2. ***Identify the domain and range of a function***
3. ***Determine the inverse of a given function***
4. ***Perform operations with functions***

3.1 Definisie van'n funksie en inleidende aspekte/ *Definition of a function and introductory aspects*

Relasies

In Wiskunde dui 'n relasie op twee versamelings (groepe getalwaardes), waartussen daar 'n verband bestaan. Vir elke element uit die een versameling kan 'n verbinding gemaak word met een of meer elemente van die ander versameling Hierdie verband is een of ander patroon of **reël**.

Hierdie reël kan 'n woordelikse instruksie wees, of 'n algebraïese formule, of 'n tabel met waardes, of 'n grafiek.

Relations

In Mathematics the word relation indicates two sets (groups of numeric values) between which there exists a relationship or connection. For each element taken from the one set a connection can be made to one or more of the elements in the other set. This relationship is usually a certain pattern or ***rule****.*

This rule may be a verbal instruction, or an algebraic formula, or a table with values, or a graph.

Voorbeeld van 'n relasie:

Voltooi die patroon:

$\left(getal; \pm\sqrt{getal}\right)$ vir die getalle $\{0; 4; 9; 16; 25\}$

Example of a relation:

Complete the pattern:

$\left(number; \pm\sqrt{number}\right)$ *for the numbers* $\{0; 4; 9; 16; 25\}$

$\{(0; ....); (4; ....); (9; \pm 3); (16; ....); (25; ....)\}$

Let daarop dat elke eerste element met twee ander elemente verbind word.

Note that each first element is connected with two other elements.

Funksies

'n Funksie is **'n reël** wat elke element van een versameling (die definisie–versameling) **verbind** met **een en slegs een element** van 'n ander versameling (die waardeversameling).

Functions

A function is **a rule** which **connects** each of the elements in the one set (the domain) with **one and only one element** in the other set (the range).

Voorbeeld van 'n funksie:

Voltooi die patroon:

$\left(getal; getal^2\right)$ vir die getalle $\{-2; -1; 0; 1; 2\}$

Example of a function:

Complete the pattern:

$\left(number; number^2\right)$ *for the numbers* $\{-2; -1; 0; 1; 2\}$

$\{(-2; ....); (-1; ....); (0; ....); (1; ....); (2; ....)\}$

3.2 Definisie en waardeversameling/ *Domain and range*

Let daarop dat elke eerste element met slegs een ander elemente verbind word.

Ons noem dit ‘n **eenduidige verband.**

Meer-eenduidige-funksies

‘n Meer-eenduidige funksie het die eienskap dat elke element uit die definisieversameling nie op ‘n unieke, verskillende element van die waardeversameling afgebeeld word nie. U sien dit duidelik by die voorbeeld hierbo oor die kwadrate. Verskillende getalle lewer dieselfde kwadrate.

Een-een-duidige funksies

‘n Een-een-duidige funksie het die eienskap dat elke element uit die definisieversameling op ‘n unieke, verskillende element van die waardeversameling afgebeeld word.

Voorbeeld

Voltooi die patroon:

$\left(getal;\ getal^3\right)$ vir die getalle $\{-2;-1;0;1;2\}$

Note that each first element is connected with only one other element.

We call this an ***unambiguous relationship.***

Many-to-one-functions

A many-to-one-function has the property that each element from the domain is not mapped unto a unique, different element of the range. You see this clearly in the example above regarding the squares. Different numbers yield the same squares.

One-to-one-functions

A one-to-one function has the property that each element from the domain is mapped unto a unique, different element of the range.

Example

Complete the pattern:

$\left(number;\ number^3\right)$ *for the numbers* $\{-2;-1;0;1;2\}$

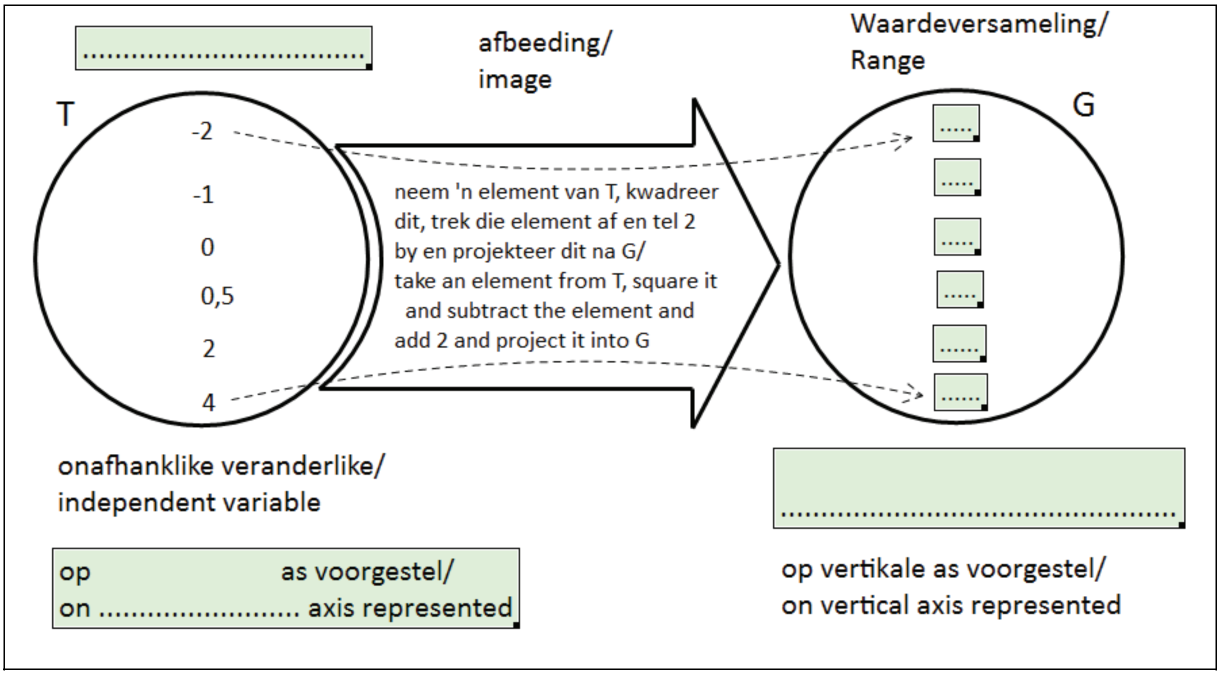
$\{(-2;\ldots\ldots\ldots);\ (-1;\ldots\ldots\ldots\ldots);(0;\ \ldots\ldots\ldots\ldots);(1;\ldots\ldots\ldots\ldots);(2;\ldots\ldots\ldots\ldots)\}$

Oefening 3.1 tot 3.2

Exercise 3.1 to 3.2

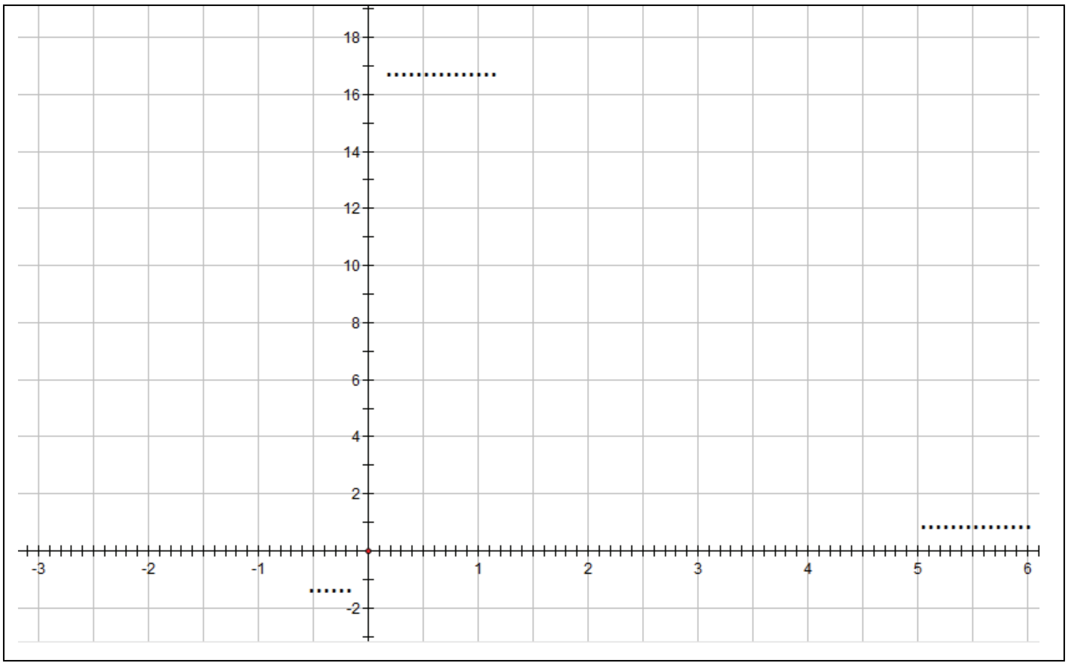
1. Beskou onderstaande skematiese voorstelling van 'n funksie en vul al die ontbrekende begrippe in.

Consider the schematic below which shows a function and fill in the ommitted concepts:



2 Stel die inligting in die skematiese voorstelling nou hieronder voor:

Now represent the information in the schematic representation below:



3 Doen **interpolasie** (trek 'n gladde kromme deur die punte wat u geplot het tussen en insluitende waar $t=-2$ en $t=4$).

*Perform **interpolation** (draw a smooth curve through the points that you just plotted beteen and including where $t=-2$ and $t=4$).*

4 Gebruik die gegewe inligting en skryf die vergelyking (formule) van die grafiek neer.

Use the given information and write down the equation (formula) of the graph.

Wenk: op skool sou dit iets gewees het soos $y=ax^2+bx+c$

Hint: at school it would have been something like $y=ax^2+bx+c$

5 Beskryf die vorm van die kromme.

Describe the shape of the curve.

stygend of dalend/ *ascending or descending*: ....
konkaafheid/ *concavity:* ....
min of maks draaipunt/ *min or max turning point*:.......................... at/*by* (.......; ....)

6 Skryf die definisieversameling van die funksie neer:

Write down the domain of the function:

$$D_g=\{\ldots\ldots|\ldots\ldots\leq\ldots\ldots\leq\ldots\ldots;\ldots\ldots\ldots\ldots\}$$

7 Skryf die waardeversameling van die funksie neer:

Write down the range of the function:

$$W_g=\{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\}$$

8 Doen nou **ekstrapolasie** (verleng die kromme "verby" die punte waar $t=-2$ en $t=4$.

*Now perform **extrapolation** (extend the curve "beyond" the points where $t=-2$ and $t=4$.*

9 Gebruik 1.4 en bereken die waardes van $g(-1{,}5)$ en $g\left(\frac{3}{2}\right)$ en $g(5)$ en $g(2+h)$

Use 1.4 and calculate the values of $g(-1{,}5)$ and $g\left(\frac{3}{2}\right)$ and $g(5)$ en $g(2+h)$

10 Dui op die grafiek aan waar u die eerste drie antwoorde op vraag 1.9 sou aflees.

Indicate on your graph where you would read off the first three answers to question 1.9.

11 Gebruik 1.4 en bereken die waardes van t sodat $g=7$.

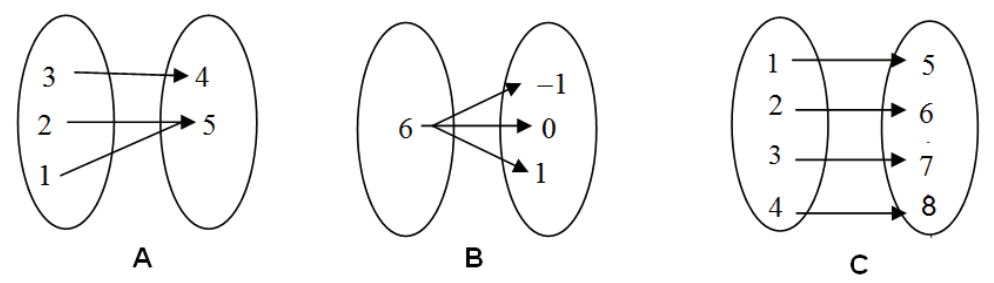
Use 1.4 and calculate the values of t such that $g=7$.

12 Dui op die grafiek aan waar u die antwoorde op vraag 1.11 sou aflees.

Indicate on your graph where you would read off the values of your answers to question 1.11.

Beskou die voorstelling:

Consider the representation:



13 Watter van die gegewe gevalle stel funksies voor?

Which of the given cases represent functions?

14 Watter van die funksies hierbo is een-tot-een-funksies?

Which of the functions above are one-to-one-functions?

15 As $A = \{-1; 0; 1; 2; 3\}$ en $B = \{-3; -2; -1; 0; 1; 2; 3; ...; 9; 10\}$, skryf die volgende funksies as versameling getallepare:

If $A = \{-1; 0; 1; 2; 3\}$ *and* $B = \{-3; -2; -1; 0; 1; 2; 3; ...; 9; 10\}$, *write the following functions as sets of ordered pairs:*

15.1 $$\{(x; y) | y = x; x \in A, y \in B\}$$

15.2 $$\{(x; y) | y = x^2; x \in A, y \in B\}$$

16 Sê of elkeen van die funksies by Vraag 15 **een-tot-een-funksies** of **meer-tot-een-funksies** is.

*Say if each of the functions in Question 15 is a **one-to-one-function** or a **many-to-one function**.*

16.1..

16.2..

16 Skets rofweg elkeen van die volgende funksies en skryf die funksie se definisieversameling en waardeversameling neer:

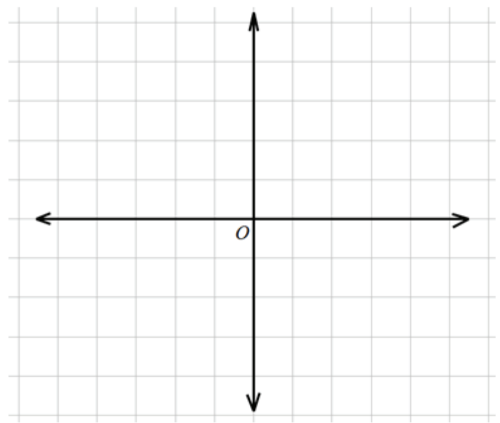
Sketch each of the following functions in rough and write down the domain and range of it:

16.1

$$\left\{(x;y)\middle|\,y=(x-1)^2-4;x\in\mathbb{R},y\in\mathbb{R}\right\}$$

$D_f = \{ \qquad\qquad \}$

$W_f = \{ \qquad\qquad \}$

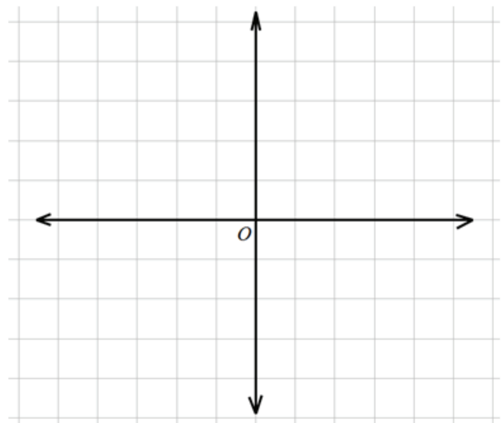


16.2

$$\left\{(t;y)\middle|\,y=-(t-2)^2+2;t\in\mathbb{R},y\in\mathbb{R}\right\}$$

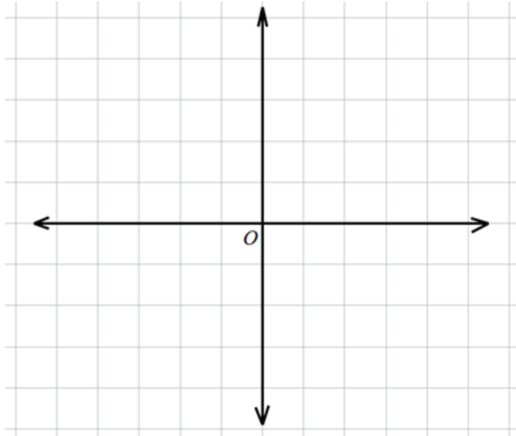
$D_f = \{ \qquad\qquad \}$

$W_f = \{ \qquad\qquad \}$

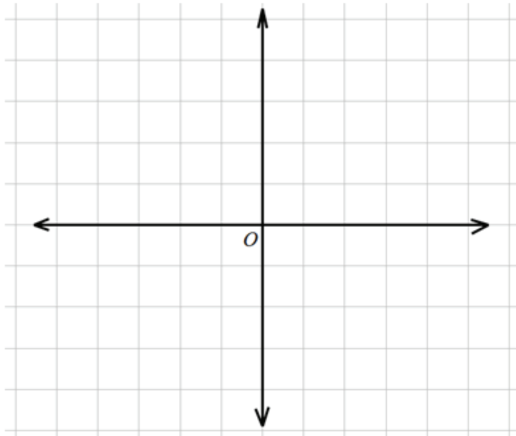


16.3

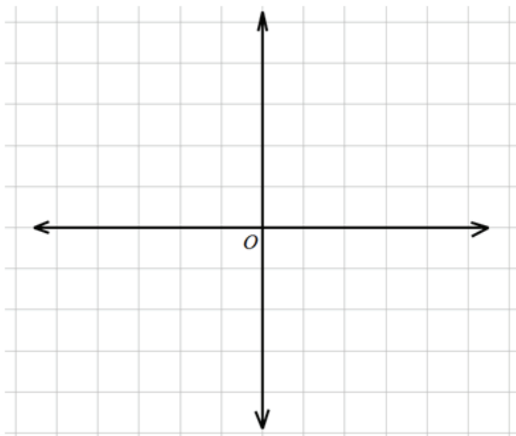
$$\left\{(z;y)\middle|\,y=-\frac{8}{z+1}+1;z\in\mathbb{R},y\in\mathbb{R}\right\}$$

$D_f = \{ \quad \}$ $W_f = \{ \quad \}$	

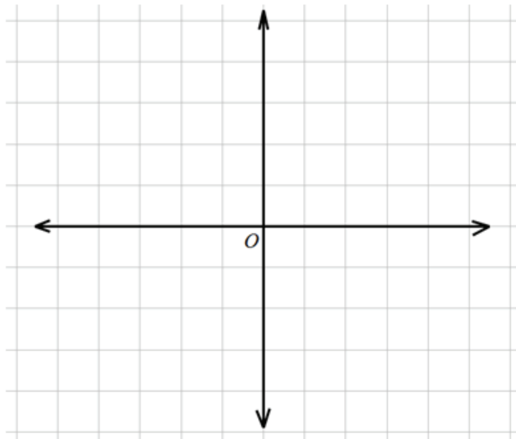
16.4 $\left\{(p;r) \middle| r=\sqrt{p-2};\, p\in\mathbb{R}, r\in\mathbb{R}\right\}$

$D_f = \{ \quad \}$ $W_f = \{ \quad \}$	

16.5 $\left\{(x;y) \middle| y=e^{x};\, x\in\mathbb{R}, y\in\mathbb{R}\right\}$

$D_f = \{ \quad \}$ $W_f = \{ \quad \}$	

16.6 $\left\{(x;y) \middle| y=\ln x;\, x\in\mathbb{R}, y\in\mathbb{R}\right\}$

$D_f = \{ \qquad \}$ $W_f = \{ \qquad \}$	

Funksie-waardes

Funksiewaardes is die afhanklike veranderlike-waardes wat op die vertikale as afgelees word.

Dit word bereken deur bepaalde waardes van die onafhanklike veranderlike in die funksie se definiërende vergelyking te vervang.

Function values

Function values are the dependent variable values which are read off the vertical axis.

They are calculated by substituting certain values of the independent variable into the defining equation of the function.

Oefening 3.2

Exercise 3.2

As/ *if* $s(t)=3$ dan is/ *then* $s(0)=$.............. en/ *and* $s(1)=$..............

en/ *and* $s(t+h)=$..............

As/ *if* $T_n = \frac{7}{2}\left(\frac{1}{2}\right)^{n-1}$ dan is/ *then* $T_1 =$ ....

....

en/ *and* $T_4 =$ ....

....

As/ *if* $g(y) = \frac{3}{2y}$ dan is/ *then* $g(0) =$ ....

en/ *and* $g(t) =$ ....

en/ *and* $g\left(\frac{2}{3}\right) =$ ....

....

....

3.3 Inverse van 'n funksie/ *Inverse of a function*

Inverse funksies

Voorbeeld

Indien $y = \sin x$ en $x = \dfrac{\pi}{6}$ dan is $y = \sin\left(\dfrac{\pi}{6}\right)$

wat $y = \dfrac{1}{2}$ lewer.

Gestel egter dat ons weet dat $y = \dfrac{1}{2}$ in $y = \sin x$ maar ons wil vir x bepaal. Dan gaan ons soos volg te werk:

$\dfrac{1}{2} = \sin x$ so $x = \sin^{-1}\left(\dfrac{1}{2}\right)$ en dit lewer onder meer $x = \dfrac{\pi}{6}$ as oplossing.

Bogenoemde illustreer die gebruik van 'n inverse funksie of inverse funksie-bewerking.

Berekening van inverse funksies

In gees en wese kom die berekening van die inverse van 'n funksie daarop neer dat...

- ons die afhanklike en onafhanklike veranderlike in die definiërende vergelyking omruil en
- dat ons die vertikale as-veranderlike (afhanklike veranderlike) weer die onderwerp van die vergelyking maak.

Daar is nietemin 'n komplikasie by die berekening van inverse funksies, naamlik dat slegs een-een-duidige funksies inverses besit.

Beskou bv. die sinus-funksie f en sy inverse f^{-1} wat deur die grafiese

Inverse functions

Example

If $y = \sin x$ *and* $x = \dfrac{\pi}{6}$ *then* $y = \sin\left(\dfrac{\pi}{6}\right)$

which yields $y = \dfrac{1}{2}$.

Suppose, however, we know that $y = \dfrac{1}{2}$ *in* $y = \sin x$ *but we wish to determine the value of* x. *Then we go about as follows:*

$\dfrac{1}{2} = \sin x$ *so* $x = \sin^{-1}\left(\dfrac{1}{2}\right)$ *and this yield, among other values,* $x = \dfrac{\pi}{6}$ *as solution.*

The argument above illustrates the use of an inverse function or inverse function operation.

Calculation of inverse functions

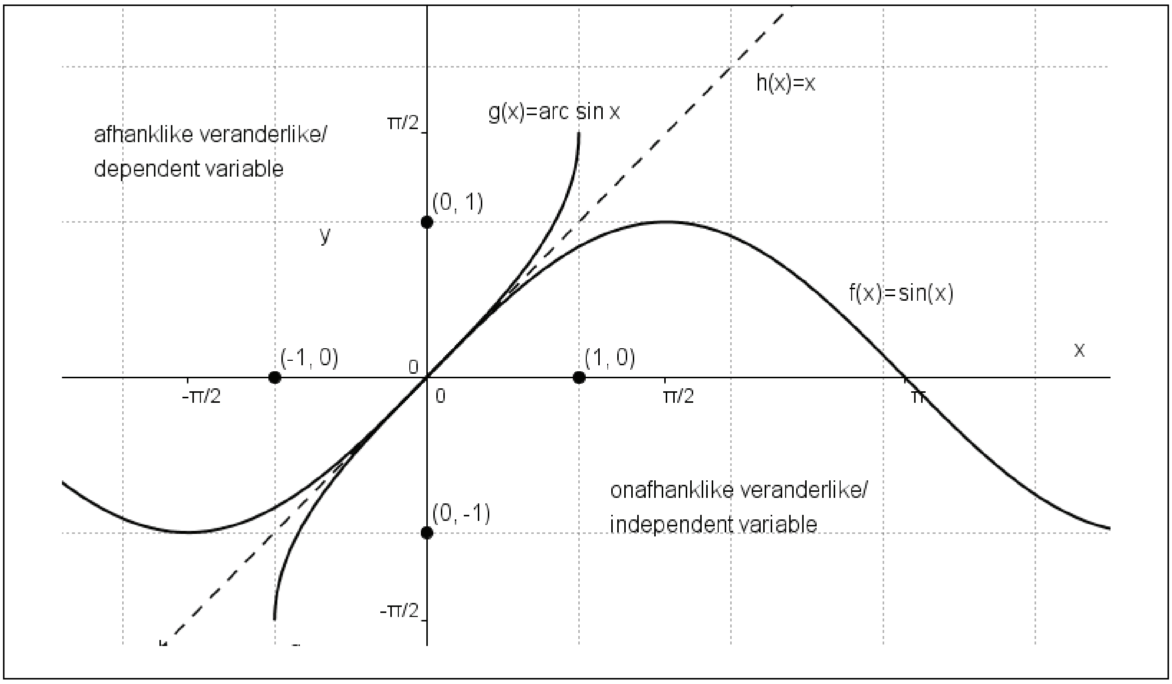
In essence the calculation of the inverse of a function boils down to...

- *that we switch the dependent and independent variable in the defining equation and*
- *that we manipulate the resulting equation to make the vertical axis-variable (dependent variable) once again the subject of the equation.*

There is, however, a complication involved with the calculation of inverse functions, and that is the fact that only one-to-one functions possess inverses.

Consider, for example, the sine function f *and its inverse* f^{-1} which is indicated by the rekenaarprogram aangedui word deur g:

graphical computer program as g:



Opmerking:

Let daarop dat ons in Leergedeelte 2.1 ‘n soortgelyke argument gevoer het om die idee van ‘n logaritmiese funksie as inverse van ‘n eksponensiële funksie in te voer.

Daar is egter sekere interessante aspekte omtrent die sinus-funksie (en ook die cosinus-funksie) wat ons noodsaak om ons diepere kennis van funksies te gebruik wanneer ons die inverse van sekere funksies bereken:

Observation:

Note that we conducted a similar argument in Study Section 2.1 where we introduced the notion of a logarithmic function as inverse of an exponential function.

There are, however, certain interesting aspects regarding the sine function (and the cosine function, too) which necessitate us to apply our deeper knowledge of functions when we calculate the inverse of certain functions:

Die implikasie van een-een-duidigheid wanneer die inverse van funksies bepaal word:

Dit is duidelik dat slegs die gedeelte van die sinus-kromme tussen die punte waar hy horisontaal loop (by sy draaipunte), ‘n spieëlbeeld in die lyn $y = x$ het.

Onthou dat ‘n funksie $y = f^{-1}(x)$ per definisie een en slegs een y-waarde besit vir elke x-waarde. Dit beteken meetkundig dat ‘n vertikale lyn wat oor die kromme van f^{-1} skuif, die kromme by elke x-waarde slegs een keer mag sny.

Maar die inverse funksie f^{-1} is uit f verkry deur die afhanklike veranderlike en die onafhanklike veranderlike te ruil – effektief het ons dus die asse geruil – en dit impliseer dat ‘n horisontale lyn wat oor f skuif, dit by elke y-waarde slegs een keer mag sny – slegs daardie deel van f se waardeversameling waarop hy die horisontale lyntoets slaag, sal die definisieversameling van f^{-1} vorm.

(Onthou dat f^{-1} in hierdie konteks NIE $\frac{1}{f}$ beteken nie.)

The implication of one-to-one-ness when we determine the inverse of a function:

It is clear that only the part of the sine curve between the points where it runs horizontally (in its turning points) possess a mirror image (reflection) in the line $y = x$ *.*

Recall that a function $y = f^{-1}(x)$ *connects by definition one and only one* y*-value with each* x*-value. Geometrically this means that a vertical line which moves over* f^{-1}*, may only intersect the curve once at every* x*-value.*

But the inverse function f^{-1} *was obtained from* f *by switching the dependent variable and the independent variable – effectively we switched the axes– and that implies that a horizontal line which moves over* f *, may intersect the curve only once at each* y*-value – only that part of the range of* f *were it passes the horizontal line test, will form the domain of* f^{-1}*.*

(Keep in mind that f^{-1} *doen NOT mean* $\frac{1}{f}$ *in this context.)*

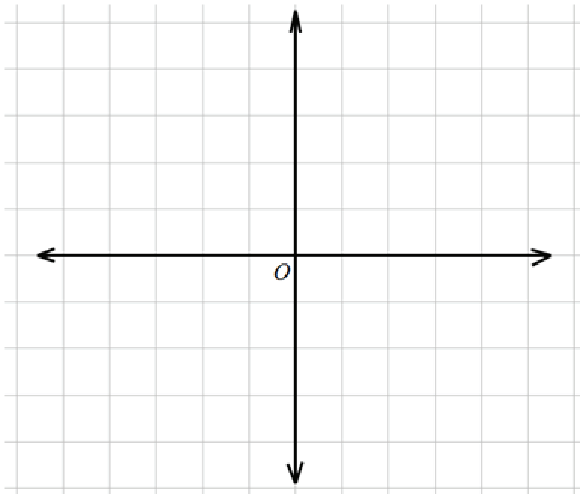
Oefening 3.3

Bepaal die inverse van die volgende funksies en skets f sowel as f^{-1}:

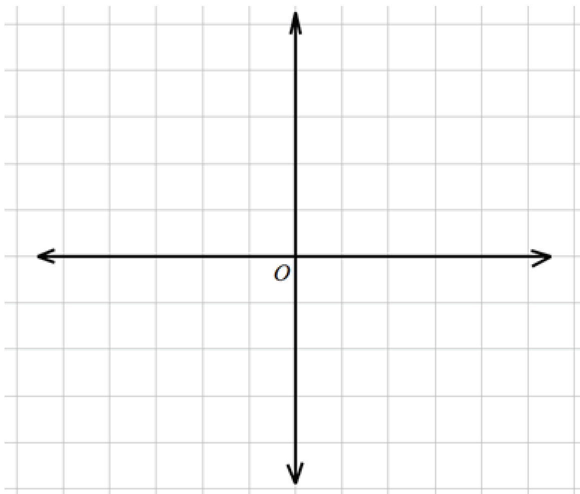
Exercise 3.3

Determine the inverse of the following functions and sketch f as well as f^{-1}:

1. $f(x) = 2x + 1$



2. $f(x) = \sqrt{x-1}$



3.4 Bewerkings met funksies / *Operations with functions*

Kombinasies van funksies (bewerkings)

Aangesien funksiewaardes getalle is, tree funksies soos getalle op waarmee ons bewerkings kan doen.

Die komplikasie, natuurlik, is die definisieversameling van die resultaat van die bewerking – **in die algemeen is die definisieversameling van die resultaat die snyding van die definisieversamelings van die afsonderlike funksies** – in die geval van deling (kwosiënte) **is die waardes van die onafhanklike veranderlike waarvoor die noemerfunksie nul is**, ook uitgesluit.

Combinations of functions (operations)

Since function values are numbers, functions behave like numbers with which we may perform operations.

The obvious complication, of course, is the domain of the result of the operation – ***in general the domain of the result will be the intersection of the domains of the separate constituent functions*** *– in the case of division (quotients)* ***the values of the independent variable for which the denominator function is zero****, will also be excluded.*

Oefening 3.4

1. As $f(x)=2x^2$ en $g(x)=3x+5$,

   bepaal die volgende

Exercise 3.4

1. If $f(x)=2x^2$ *and* $g(x)=3x+5$,

   determine the following

1.1 $(f+g)(x)=f(x)+g(x)=$.................... + ...

....

1.2 $(g+f)(x)=$................ + ... = ... + ...

....

1.3 $(f-g)(x)=$................ − .... = ... − ....

....

1.4 $(g-f)(x)=$................ − .... = ... − ....

....

1.5 $(fg)(x)=$................ × .... = ... · ....

....

1.6 $(gf)(x) = \dots\dots\dots\dots \times \dots\dots\dots\dots = \dots\dots\dots\dots \times \dots\dots\dots\dots$

$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$

1.7 $\left(\frac{f}{g}\right)(x) =$ ________________ $=$ ____________________________

$=$ ____

$=\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$

1.8 $\left(\frac{g}{f}\right)(x) =$ ________________ $=$ ____________________________

$=$ ____

$=\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$

2. Watter van die bewerkings in hierbo is kommutatief?

2. Which ones of the operations above are commutative?

3. As $f(x)=2x^2$, $a=3$ en $x=2$ bepaal:

3. If $f(x)=2x^2$, $a=3$ and $x=2$ determine:

3.1 $a\left[f(x)\right]=$ ....

....

....

3.2 $f(ax)=$ ....

....

....

3.3 Watter gevolgtrekking kan u maak oor 3.1 en 3.2/

3.3 *Which conclusion do you make as a result of 3.1 and 3.2?*

4. As $g(x)=\sqrt{x}$, $a=9$ en $x=16$ bepaal:

4. If $g(x)=2x^2$, $a=9$ and $x=16$ determine:

4.1 $g(a+x)=$ ....

....

....

4.2 $g(a)+g(x)=$ ………………………………………………………………………….

………………………………………………………………………

…………………………………………………………………………..

4.3 Watter gevolgtrekking kan u maak oor 4.1 en 4.2?

3.4 *Which conclusion do you make as a result of 4.1 and 4.2?*

3.5 Saamgestelde funksies/ *Composite functions*

Saamgestelde funksies

'n Saamgestelde funksie kan beskou word as 'n funksie binne-in 'n ander funksie – soms word daar van "geneste funksies" gepraat. Wanneer so 'n funksie se waarde bepaal word, werk ons van binne na buite.

Voorbeeld 1

Die volume van 'n sferiese ballon word gegee deur die formule $V(r)=\frac{4}{3}\pi r^3$. Die radius van die ballon verander egter met tyd volgens die formule $r(t)=-t^2+6t$ met $0\leq t\leq 3$ waar radius in cm gemeet word en tyd in sekondes.

Ons wil die volume bereken op die oomblik wanneer $t=\frac{3}{2}$.

Nou kan ons die volume skryf as

Composite functions

A composite function may be considered as a function within another function – some writers refer to this phenomenon as "nested functions". Whenever we evaluate such a function, we proceed from the inside out.

Example 1

The volume of a spherical balloon is given by the formula $V(r)=\frac{4}{3}\pi r^3$. The radius changes with time according to the formula $r(t)=-t^2+6t$ with $0\leq t\leq 3$ where the radius is measured in cm and the time in seconds.

We wish to calculate the volume at the instant when $t=\frac{3}{2}$.

Now we may write the volume as

$$(V\circ r)(t)=V(r(t))=\underbrace{\frac{4}{3}\pi\left(\underbrace{-t^2+6t}_{r(t)}\right)^3}_{V(r)}$$

Ons kan die uitdrukking regs probeer vereenvoudig deur die hakies uit te vermenigvuldige totat ons die volume as 'n veeltermfunksie $V(t)$ het – daarna kan ons die waarde van $t=\frac{3}{2}$ in vervang om $V\left(\frac{3}{2}\right)$

We could attempt to simplify the expression right by multiplying out the parentheses until we obtain the volume as a polynomial function $V(t)$ – then we could simply substitute $t=\frac{3}{2}$ in order to obtain $V\left(\frac{3}{2}\right)$.

te verkry.

Dit sou egter 'n groot klomp rekenwerk afgee.

Die teorie van saamgestelde funksies laat ons toe om eerder soos volg te werk te gaan:

Bereken $r\left(\frac{3}{2}\right)$ en vervang die antwoord in die formule $V(r)=\frac{4}{3}\pi r^3$; dan verkry ons ook die waarde van V as $t=\frac{3}{2}$.

This approach, however, would generate a lot of tedious calculation.

The theory of composite functions permits us to rather proceed as follows:

Compute $r\left(\frac{3}{2}\right)$ *and substitute the answer into* $V(r)=\frac{4}{3}\pi r^3$*; then we also obtain the value of* V *when* $t=\frac{3}{2}$.

Voltooi/ Complete:

$r\left(\frac{3}{2}\right)$=………………………………………………

=………………………………………………

$V(......)=\frac{4}{3}\pi(..............)^3$

=……………………………………………..

=………………………………………………

Wat is die definisieversameling van r ?

What is the domain of r *?*

Wat is die waardeversameling van r ?

What is the range of r *?*

Wat is die definisieversameling van V ?

What is the domain of V *?*

Wat is die waardeversameling van V ?

What is the range of V *?*

Voorbeeld 2 | ***Example 2***

Gegee: $A(r)=4\pi r^2$

$r(t)=t^2+5t+5$ met $0\le t\le 10$

Given: $A(r)=4\pi r^2$

$r(t)=t^2+5t+5$ *with* $0\le t\le 10$

Ons wil A bereken vir $t=6$.

We wish to calculate A for $t=6$.

Voltooi:

Complete:

$$(A\circ r)(t)=\ldots\ldots\ldots\ldots=\underbrace{4\pi\left(\underbrace{\ldots\ldots\ldots\ldots\ldots}_{r(t)}\right)^2}_{A(r)}$$

$r(6)=$..

=..

= $A(......)=4\pi(..............)^3$

=..

=..

Wat is die definisieversameling van r ?

What is the domain of r ?

Wat is die waardeversameling van r ?

What is the range of r ?

Wat is die definisieversameling van A ?

What is the domain of A ?

Wat is die waardeversameling van A ?

What is the range of A ?

In die algemeen definieer ons 'n saamgestelde funksie soos volg:

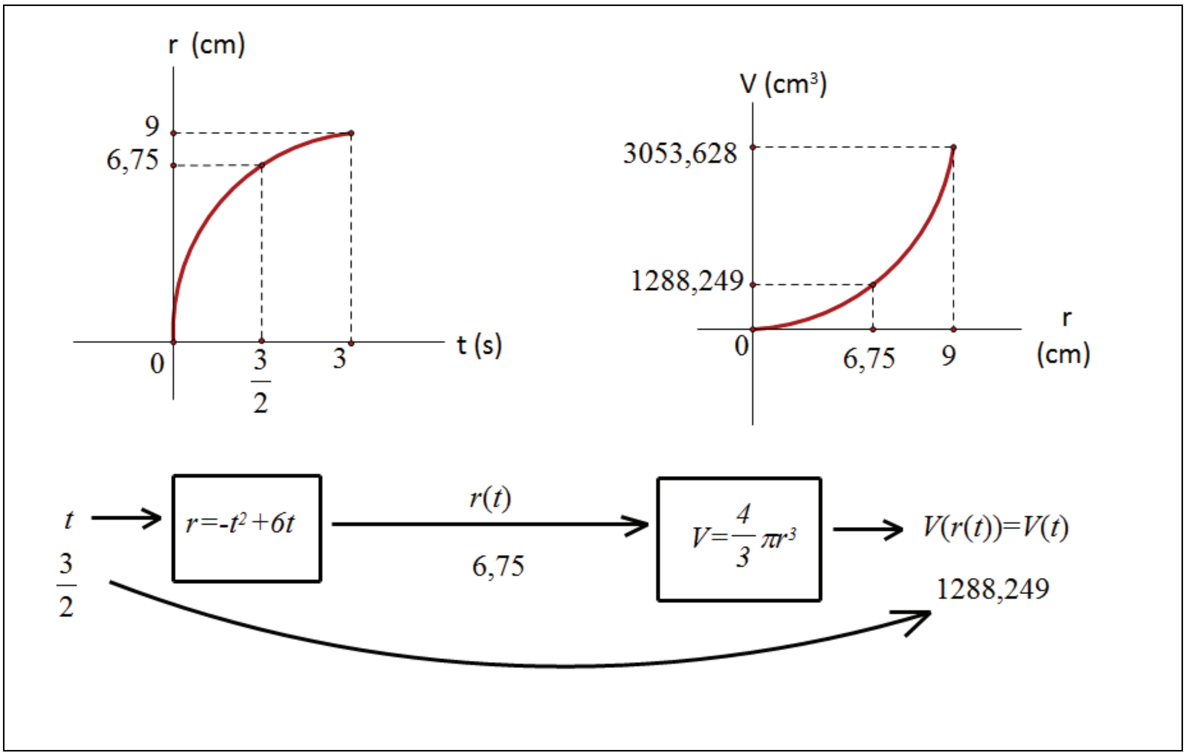
In general we define a composite function as follows:

As f en g funksies van x is, dan is die saamgestelde funksie $(f \circ g)(x)$ die geneste funksie $f(g(x))$ vir alle x in die definisieversameling van g sodat $g(x)$ die definisieversameling van f vorm.

If f and g are functions of x, then the composite function $(f \circ g)(x)$ is the nested function $f(g(x))$ for all x in the domain of g in order that $g(x)$ forms the domain of f.

Skematies kan ons die voorbeeld oor die ballon (Voorbeeld 1 hierbo) soos volg voorstel:

Schematically we can represent the example about the balloon (Example 1 above) as follows:



By die **kettingreël vir differensiasie** wat u later vanjaar volledig behandel, sal u in staat moet wees om 'n saamgestelde funksie se binneste en buitenste deel te

In the application of the ***chain rule for derivatives*** *which you will study in detail later this year, you must be able to identify the inner and outer constituents of a composite* identifiseer. *function.*

Gewoonlik word die simbool u of g vir die binneste funksie gebruik en die simbool v of f vir die buitenste funksie.

Usually the symbol u or g is used for the inner function and the symbol v or f is used for the outer function.

Voorbeeld 1 ***Example 1***

As/ *if* $f(x)=x^2$ en/*and* $g(x)=\sqrt{x+2}$ dan is/ *then:*

$(f\circ g)(x)=f(g(x))=\left[\sqrt{x+2}\right]^2$ en dit lewer/ *this yields* $(f\circ g)(x)=x+2$

En/*and*

$(g\circ f)(x)=g(f(x))=\sqrt{(x^2)+2}$ en dit lewer/ *this yields* $(g\circ f)(x)=\sqrt{x^2+2}$

Voorbeeld 2 ***Example 2***

Ontbind in die funksies/ *Resolve into into the functions* $u(x)$ en/*and* $v(u)$ as/ *if*

$$f(x)=\cos\left(\frac{1}{x^2-x+1}\right)$$

Oplossing/*Solution*

$$f(x)=\underbrace{\cos\underbrace{\left(\frac{1}{x^2-x+1}\right)}_{u(x)}}_{v(u)}$$

Dus/ *So* $u(x)=\dfrac{1}{x^2-x+1}$ en/ *and* $v(u)=\cos u$

Oefening 3.5 ***Exercise 3.5***

1. As $v(t)=\sqrt[3]{t}$ en $u(t)=\sin t$, bepaal

1. If $v(t)=\sqrt[3]{t}$ and $u(t)=\sin t$, die volgende: determine:

1.1 $(v \circ u)(t)$

1.2 $(u \circ v)(t)$

2. As $v(t)=\dfrac{2}{1+t}$ en $u(t)=\dfrac{3}{1-t}$, bepaal die volgende:

2. If $v(t)=\dfrac{2}{1+t}$ and $u(t)=\dfrac{3}{1-t}$, determine:

2.1 $(v \circ u)(t)$

2.2 $(u \circ v)(t)$

Blaai na die bylae van hierdie boek en voltooi Werkkaart 3.

Turn to the addendum of this book and complete Worksheet 3.

4 Radiaalmaat en trigonometrie/ *Radian measure and trigonometry*

Leerdoelstellings vir hierdie leereenheid	*Learning aims for this study unit*
Na afhandeling van hierdie leereenheid moet die student in staat wees om die volgende te doen: 1. **Radiaalmaat te definieer en hoeke van grade na radiale om te skakel en andersom** 2. **Booglengte te bereken** 3. **Oppervlakte van 'n sirkelsektor te bereken** 4. **Al ses trigonometriese verhoudings te definieer en hul funksiewaardes in al vier kwadrante van die platvlak te bereken** 5. **Die som- en verskilformules toe te pas** 6. **Die dubbelhoekformules toe te pas** 7. **Trigonometriese indentiteite te bewys**	***Upon completion of this study unit the student must be able to do the following:*** 1. ***Define radian measure and convert angles form degrees to radians and vice versa*** 2. ***Calculate arc length*** 3. ***Calculate the area of a circle sector*** 4. ***Define all six trigonometric ratios and calculate their function values in all four quadrants of the flat plane*** 5. ***Apply the sum and difference formulas*** 6. ***Apply the double angle formulas*** 7. ***Prove trigonometric identities***

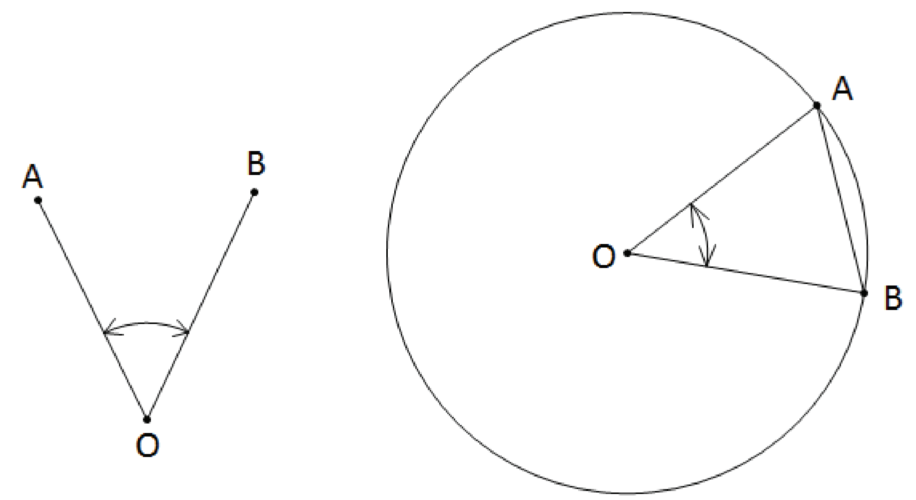
4.1 Radiaalmaat/ *Radian measure*

Die verdeling van 'n sirkel in 360 gelyke dele wat ons grade noem, asook die onderverdeling van 'n graad in 60 gelyke dele wat ons minute noem, is 'n antieke gebruik wat by die Babiloniese en Sumeriese geleerdes van voor 540 vC oorgeneem is.

The subdivision of a circle into 360 equal parts which we call degrees, as well as the subdivision of a degree into 60 equal parts which we call minutes, is an ancient tradition which were passed on from the Babylonian and Sumerian scholars of earlier than 540 B.C.

Probeer gerus met behulp van die volgende sketse verduidelik wat ons bedoel met die begrip "hoek".

Use the following sketches and attempt to explain what we mean by the concept "angle".

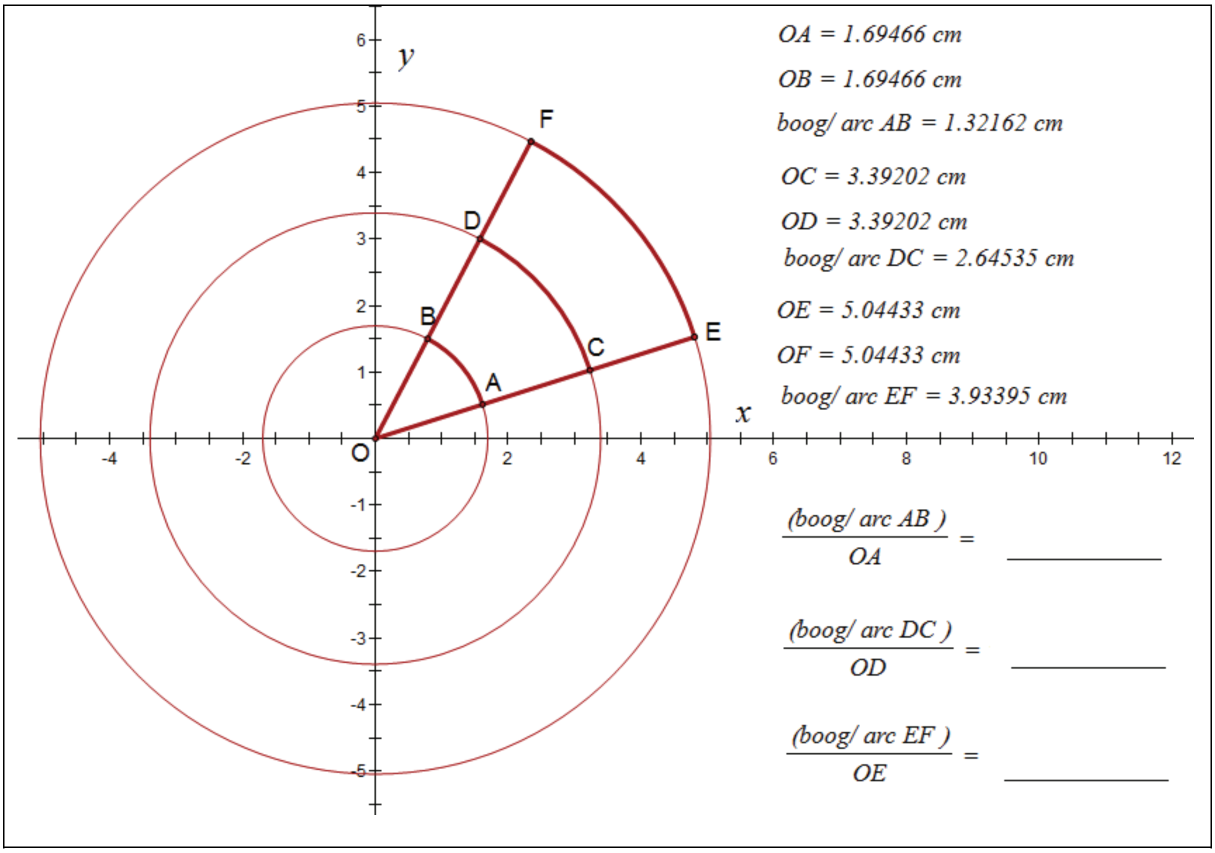


Vir moderne Wiskunde en wetenskap vereis ons 'n formele definisie vir die begrip **hoek**.

*For modern Mathematics and science we require a formal definition for the concept **angle**.*

Beskou onderstaande skets waarop akkurate metings aangetoon word en bereken vir al drie sektore die waarde van die verhouding $\frac{booglengte}{radius}$:

Consider the sketch below where accurate measurements are shown for all three sectors in order that the ratio $\frac{arc\ length}{radius}$ *may be calculated:*



Die verhouding $\frac{booglengte}{radius}$ bly konstant in al drie gevalle, ongeag die grootte van die oppervlakte van die gebied tussen die lyne en/of boë wat die sektore omsluit.

Daarom is dit nuttig om die hoek tussen enige twee radiusse voortaan eenvoudig te definieer as $\theta = \frac{booglengte}{radius}$.

Let daarop dat die hoek θ geen eenheid besit nie; hoekom?

Ons definieer daarom eenheid van 'n hoek θ wat gedefineer is as $\theta = \frac{booglengte}{radius}$ as radiale.

Hoe groot is 'n radiaal en hoeveel radiale pas in 'n volsirkel, wat ons tradisioneel beskou as die boog van een omwenteling (360°)?

Beskou die volgende skets en voltooi die ontbrekende inligting:

Note that the ratio $\frac{arc\ length}{radius}$ remains constant in all three cases, irrespective of the area between the lines and/ or the arcs defining the sectors.

That is why it is useful to henceforth simply define the angle between any two radii from now on as $\theta = \frac{arc\ length}{radius}$.

Note that the angle θ does not possess any units; why?

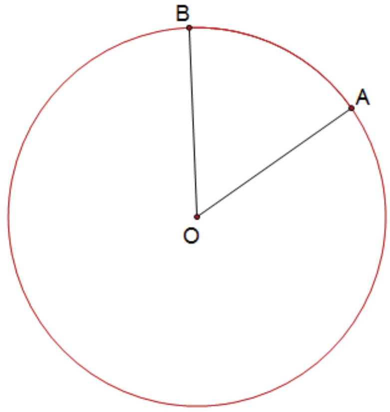
Therefore we define the unit of an angle θ which is defined as $\theta = \frac{arc\ length}{radius}$ radians.

How big is one radian and how many radians fit into a full circle, which we traditionally consider as the arc of one revolution (360°)?

Consider the following sketch and complete the information on it:

boog/ arc AB = 3.206 cm

radius = 3.206 cm



$$\frac{\text{boog/ arc AB}}{\text{radius}} = \underline{\qquad\qquad\qquad}$$

U kan sien dat 'n hoek van 1 radiaal 'n groot hoek is; skat gerus hoe groot die hoek hierbo in grade sou wees.

You can see that an angle of 1 radian is a rather large angle; estimate the size of the angle in degrees.

Laat ons nou 'n metode ontwikkel om radiale in grade om te skakel:

Let us now develop a method to convert radians to degrees:

Beskou 'n sirkel met radius r en sirkelboog s waar die sirkelboog die hele omtrek van die sirkel is. In hierdie geval is die hoek wat deur twee radiusse en die boog (dit is nie omtrek) onderspan word, tog een omwenteling, dit is 360° volgens ons tradisionele hoekmaat.

Consider a circle with radius r and circle arc s where the arc is in this case the entire circumference of the circle. In this case the angle which is subtended by the radii and the arc (the circumference) is one revolution, which is 360° in terms of traditional units.

Voltooi nou die redenasie:

Complete the reasoning:

$$\text{Een omwent}\left(^\circ\right) = 1\,\text{Omwent}\left(\text{radiale}\right)$$

$$\therefore 360^\circ = \frac{booglengte\,(volsirkel)}{radius} \quad (\text{per def.})$$

$$\therefore 360^\circ = \frac{\dots\dots\dots\dots\dots\dots}{r} \quad (\text{omtrek=?})$$

$$\therefore 360^\circ = \dots\dots\dots\dots\dots \quad \text{radiale}$$

$$\therefore 1^\circ = \dots\dots\dots\dots\dots \quad \text{radiale}$$

$$1\,\text{radiaal} = \dots\dots\dots\dots^\circ$$

$$\text{One rev}\left(^\circ\right) = 1\,\text{Rev}\left(\text{radians}\right)$$

$$\therefore 360^\circ = \frac{arc\ length\,(full\ circle)}{radius} \quad (\text{by def.})$$

$$\therefore 360^\circ = \frac{\dots\dots\dots\dots\dots\dots}{r} \quad (\text{circumf=?})$$

$$\therefore 360^\circ = \dots\dots\dots\dots\dots \quad \text{radians}$$

$$\therefore 1^\circ = \dots\dots\dots\dots\dots \quad \text{radians}$$

$$1\,\text{radian} = \dots\dots\dots\dots^\circ$$

Dit is nou maklik om te sien dat ons 'n hoek van π radiale as 'n hoek van $180°$ kan beskou.

Omskakelings soos die volgende volg dan maklik:

It is now easy to see why we may consider an angle of π radians *as an angle of* $180°$.

Conversions such as the following are now simple:

Voorbeeld/ *Example*

1 Skakel 240° om na radiale/ *Convert 240° to radians*

2. Skakel 315° om na radiale/ *Convert 315° to radians*

3. Skakel $\frac{3}{4}\pi$ radiale om na grade/ *Convert* $\frac{3}{4}\pi$ *radians to degrees*

4. Skakel $\frac{11}{6}\pi$ radiale om na grade/ *Convert* $\frac{11}{6}\pi$ *radians to degrees*

Oplossing/ *Solution*

1.
$$\begin{aligned} 240° &= 180° + 60° \\ &= \pi \ rad + \frac{1}{3}\pi \ rad \\ &= \frac{3}{3}\pi \ rad + \frac{1}{3}\pi \ rad \\ &= \frac{4}{3}\pi \ rad \end{aligned}$$

2.
$$\begin{aligned} 315° &= 360° - 45° \\ &= 2\pi \ rad - \frac{1}{4}\pi \ rad \\ &= \frac{8}{4}\pi \ rad - \frac{1}{4}\pi \ rad \\ &= \frac{7}{4}\pi \ rad \end{aligned}$$

3.
$$\begin{aligned} \frac{3}{4}\pi \ rad &= \frac{3}{4}\times 180° \\ &= 135° \end{aligned}$$

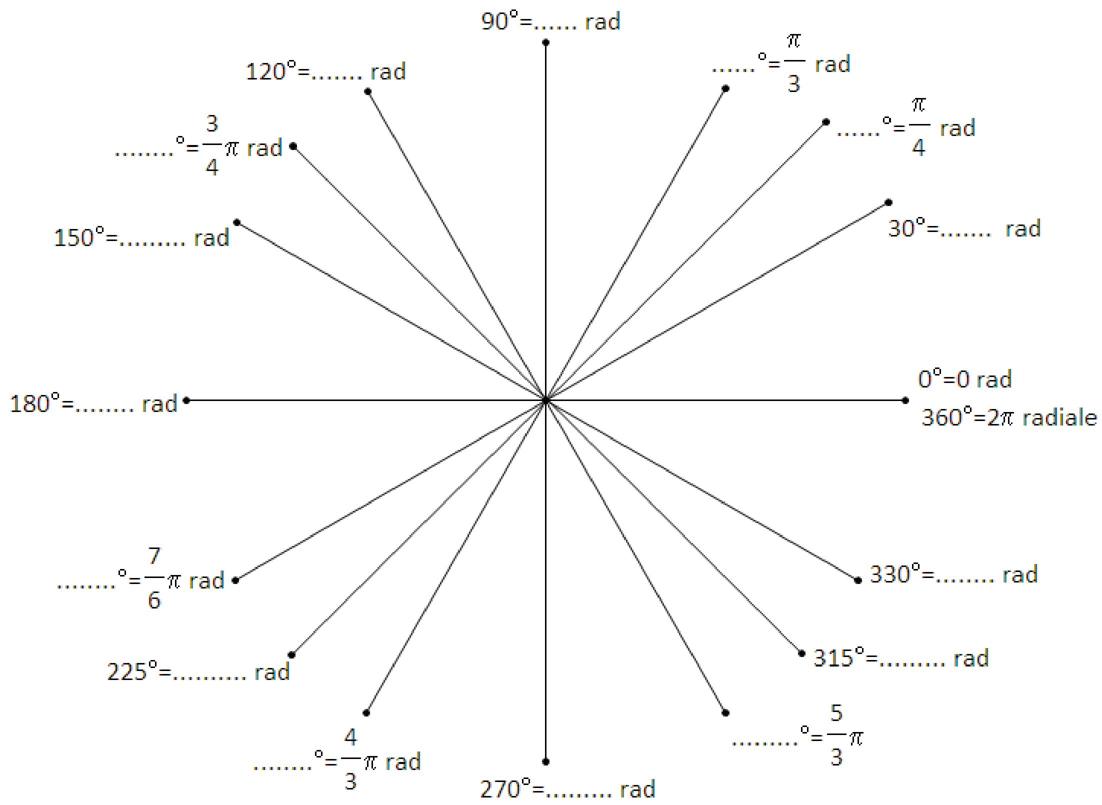
4.
$$\begin{aligned} \frac{11}{6}\pi \ rad &= \frac{11}{6}\times 180° \\ &= 330° \end{aligned}$$

Oefening 4.1

Voltooi nou die ontbrekende inligting in onderstaande skets:

Exercise 4.1

Complete the omitted information in the sketch below:



4.2 Berekening van booglengte/ *Calculation of arc length*

Uit die definisie $\theta \text{ in rad} = \dfrac{booglengte}{radius}$ volg nou die volgende interessante toepassings:

From the definition $\theta \text{ in rad} = \dfrac{arc\ length}{radius}$ *follows interesting applications:*

Oefening 4.2

1. 'n Speelgoedtrein ry in 'n sirkelbaan met 'n oppervlakte van $25\,446{,}900\,49 \text{ cm}^2$. Indien dit 2,5 m aflê tussen twee punte op die spoor, wat is die hoek in grade waardeur die treintjie gery het?

Exercise 4.2

1. *A toy train travels around a circular track with an area of* $25\,446{,}900\,49 \text{ cm}^2$. *If it covers* 2,5 m *between two points on the track, what is the angle in degrees through which the train travelled?*

2. Gestel 'n sirkelsektor het 'n radius van x en 'n middelpuntshoek van 2 radiale. Bereken die lengte van die boog van die sektor.

2. *Suppose a circle sector has a radius of* x *and a central angle of* 2 radians. *Calculate the length of the arc of the sector.*

3. Bepaal die oppervlakte van 'n sirkel indien 'n booglengte van $200\,\text{mm}$ onderspan word deur 'n hoek van $171{,}887\,339^\circ$.

3. Determine the area of a circle if an arc of $200\,\text{mm}$ *is subtended by an angle of* $171{,}887\,339^\circ$.

4.3 Berekening van die oppervlakte van 'n sirkelsektor/ *Calculation of the area of a circle sector*

Beskou die volgende tabel en kyk of u die patroon kan raaksien:

Consider the following table and see if you can discover the pattern:

Figuur/ *Figure*	Oppervlakte-Formule/ *Area formula*	Middelpunts-hoek in rad/ *Angle in rad*
Volsirkel/ *Full circle*	$A = \pi r^2$	2π
Halfsirkel/ *Semi-circle*	$A = \ldots\ldots\ldots\ldots \pi r^2$	π
Kwartsirkel/ *Quarter circle*	$A = \ldots\ldots\ldots\ldots \pi r^2$	....
Agstesirkel/ *Eighth of circle*	$A = \frac{1}{8}\pi r^2$	$\frac{\pi}{4}$
n de van 'n sirkel/ n *th of a circle*	$A = \ldots\ldots\ldots\ldots \pi r^2$	....

Voltooi: Indien 'n sirkelsektor 'n hoek van θ radiale by die middelpunt maak, is die formule vir die oppervlakte van die sektor:

$A = \ldots\ldots\ldots\ldots r^2$

Complete: If a circle sector makes an angle of θ radians *at the centre, the formula for the area of the sector is:*

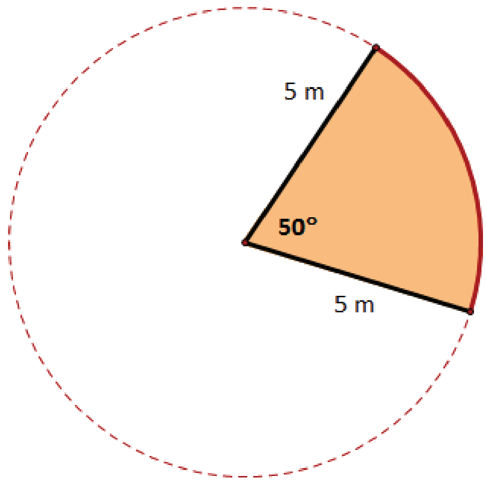
$A = \ldots\ldots\ldots\ldots r^2$

Oefening 4.3

1. Die straal van 'n waterspreier beweeg deur 'n hoek van 50° en die straal kan $5\,\text{m}$ ver bykom. Bereken die oppervlakte wat dit kan natlei.

Exercise 4.3

1. *The jet of a water sprayer sweeps through an angle of 50° and the jet can reach $5\,\text{m}$ far. Calculate the area which the sprayer can irrigate.*



2. Gestel die hoek θ waardeur die straal beweeg, verdubbel maar die radius van die straal halveer. Met watter persentasie sal die oppervlakte wat besproei kan word, dan verander?

   Is dit 'n toename of afname?

2. *Suppose the angle θ through which the jet sweeps, doubles but the radius halves. By which percentage would the area which can be irrigated, then change?*

   Is this an increase or decrease?

3. Gestel die hoek θ waardeur die straal beweeg, halveer maar die radius van die straal verdubbel. Met watter persentasie sal die oppervlakte wat besproei kan word, dan verander?

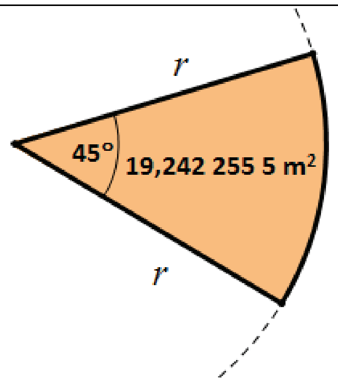
   Is dit 'n toename of afname?

3. Suppose the angle θ through which the jet sweeps, halves but the radius doubles. By which percentage would the area which can be irrigated, then change?

Is this an increase or decrease?

4. Bepaal die radius:

4. Bepaal die radius:



NWU®

4.4 Die ses trigonometriese verhoudings en hul funksiewaardes in al vier kwadrante van die platvlak/ *The six trigonometric ratios and their function values in all four quadrants of the flat plane*

Beskou die reghoekige driehoek: *Consider the right triangle*

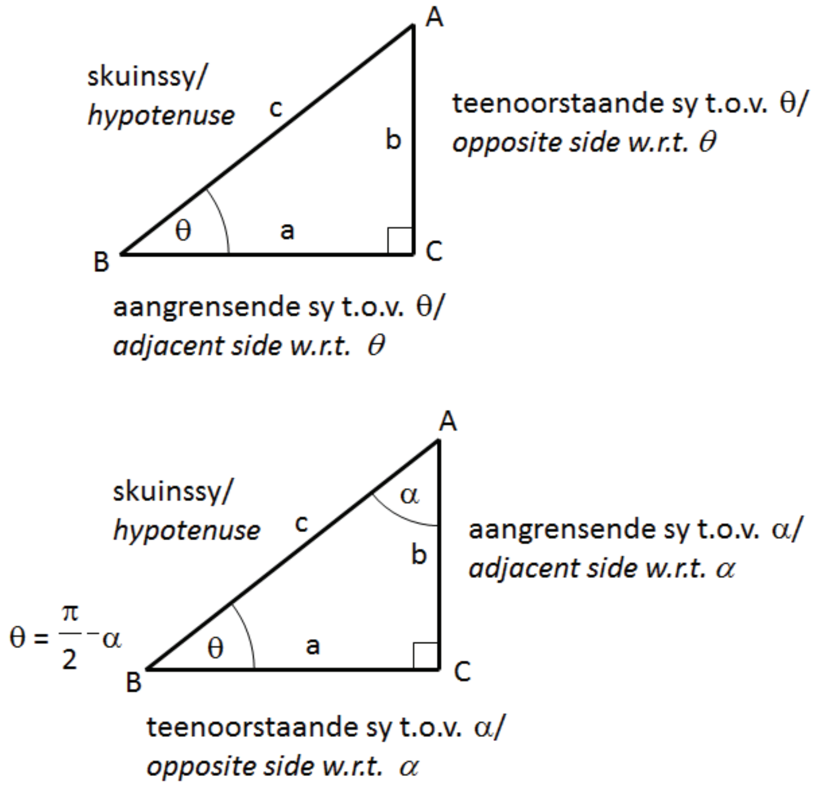


Fig. 4.4.1

Aangesien daar ses verskillende maniere bestaan om die teenoorstaande sy, aangrensende sy en skuinssy van 'n regte driehoek as verhoudings van mekaar te skryf, geld nie net die volgende drie verhoudings

Since there exist six different ways in which to write ratios for the opposite side, anjeacent side and hypotenuse of a right triangle, not only the following ratios hold true

$\sin\theta = \dfrac{b}{c}$	en/and	$\sin\alpha = \dfrac{\ldots\ldots\ldots}{\ldots\ldots\ldots}$
$\cos\theta = \dfrac{\ldots\ldots\ldots}{\ldots\ldots\ldots}$	en/and	$\cos\alpha = \dfrac{\ldots\ldots\ldots}{c}$
$\tan\theta = \dfrac{\ldots\ldots\ldots}{\ldots\ldots\ldots}$	en/and	$\tan\alpha = \dfrac{\ldots\ldots\ldots}{\ldots\ldots\ldots}$

nie, maar ook die volgende verhoudings

but also the following ratios

$\operatorname{cosec}\theta = \dfrac{1}{\sin\theta} = \dfrac{c}{b}$	en/and	$\operatorname{cosec}\alpha = \dfrac{1}{\sin\alpha} = \dfrac{\ldots\ldots\ldots}{\ldots\ldots\ldots}$
$\sec\theta = \dfrac{1}{\cos\theta} = \dfrac{\ldots\ldots\ldots}{\ldots\ldots\ldots}$	en/and	$\sec\alpha = \dfrac{1}{\cos\alpha} = \dfrac{\ldots\ldots\ldots}{\ldots\ldots\ldots}$
$\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{\ldots\ldots\ldots}{\ldots\ldots\ldots}$	en/and	$\cot\alpha = \dfrac{1}{\tan\alpha} = \dfrac{\ldots\ldots\ldots}{\ldots\ldots\ldots}$

Die verhoudings in die tweede groep hierbo staan as **resiprookverhoudings of omgekeerde verhoudings van die sinus-, cosinus- en tangensverhoudings** bekend.

(NIE INVERSE VERHOUDINGS NIE)

Dit kan maklik aangetoon word dat $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ deur van die eerste skets in Fig. 4.4.1 gebruik te maak:

The ratios in the second group are known as **reciprocal ratios of the sine, cosine and tangent ratios**.

(NOT INVERSE RATIOS)

IT can easily be shown that $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ by using the first sketch in Fig. 4.4.1:

Voorbeeld/ *Example*

1. Bewys dat $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

Oplossing/ *Solution*

Linkerkant/ $LHS = \tan\theta$

$$= \frac{b}{a}$$

Regterkant/ $RHS = \dfrac{\sin\theta}{\cos\theta}$

$$= \frac{\left(\frac{b}{c}\right)}{\left(\frac{a}{c}\right)}$$

$$= \frac{b}{\cancel{c}} \times \frac{\cancel{c}}{a}$$

$$= \frac{b}{a}$$

$\therefore LHS = RHS$ dus/*so* $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

Maak nou van die tweede skets in Fig. 4.4.1 gebruik en bewys self dat $\cot\alpha = \dfrac{\cos\alpha}{\sin\alpha}$.

Now make use the second sketch in Fig 4.4.1 and prove on your own that $\cot\alpha = \dfrac{\cos\alpha}{\sin\alpha}$

Die tweede skets in Fig. 4.4.1 gee ook aan ons 'n manier om die **ko-verhoudings van die ses trigonometriese verhoudings** verkry.

The second sketch in Fig. .4.4.1 also provides us with a way to obtain ***the co-ratios of the six trigonometric ratios****.*

Voltooi die volgende:

Complete the following:

$\sin\alpha = \dfrac{\ldots\ldots\ldots\ldots}{\ldots\ldots\ldots\ldots}$ en/ *and* $\cos\theta = \dfrac{\ldots\ldots\ldots\ldots}{\ldots\ldots\ldots\ldots}$ dus/ *so*$\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots$

maar aangesien/ *but because* $\theta = \dfrac{\pi}{2} - \alpha$ kan ons skryf/ *we may write*

$\sin\ldots\ldots\ldots = \cos\left(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\right)$

$\cos\alpha = \dfrac{\ldots\ldots\ldots\ldots}{\ldots\ldots\ldots\ldots}$ en/ *and* $\sin\theta = \dfrac{\ldots\ldots\ldots\ldots}{\ldots\ldots\ldots\ldots}$ dus/ *so*$\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots$

maar aangesien/ *but because* $\theta = \dfrac{\pi}{2} - \alpha$ kan ons skryf/ *we may write*

$\cos\ldots\ldots\ldots = \sin\left(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\right)$

$\tan\alpha = \dfrac{\ldots\ldots\ldots\ldots}{\ldots\ldots\ldots\ldots}$ en/ *and* $\cot\theta = \dfrac{\ldots\ldots\ldots\ldots}{\ldots\ldots\ldots\ldots}$ dus/ *so*$\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots$

maar aangesien/ *but because* $\theta = \dfrac{\pi}{2} - \alpha$ kan ons skryf/ *we may write*

$\tan\ldots\ldots\ldots = \cot\left(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\right)$

$\sec\alpha = \dfrac{\ldots\ldots\ldots\ldots}{\ldots\ldots\ldots\ldots}$ en/ *and* $\operatorname{cosec}\theta = \dfrac{\ldots\ldots\ldots\ldots}{\ldots\ldots\ldots\ldots}$ dus/ *so*$\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots$

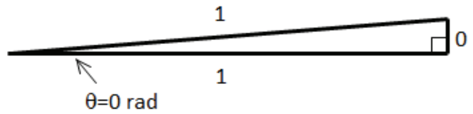
maar aangesien/ *but because* $\theta = \dfrac{\pi}{2} - \alpha$ kan ons skryf/ *we may write*

$\sec\ldots\ldots\ldots = \operatorname{cosec}\left(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\right)$

Dit is ook aan u bekend dat die trigonometriese funksiewaardes vir **sekere bekende hoeke**, nl. $0°, 30°, 45°, 60°$ en $90°$ baie maklik uit die volgende sketse bepaal en ook maklik gememoriseer kan word; Ons pas onderstaande sketse vir radiaalmaat aan.

It is also known to you that the function values for ***certain familiar angles,*** *namely.* $0°, 30°, 45°, 60°$ and $90°$ *may very easily be obtained from the following sketches and may also be easily memorized; We adjusted the sketches below for radian measure.*

Voltooi die ontbrekende inligting: *Complete the omitted information:*



$\sin 0 =$ $\quad$ $\operatorname{cosec} 0 =$

$\cos 0 =$ $\quad$ $\sec 0 =$

$\tan 0 =$ $\quad$ $\cot 0 =$

$\sin \frac{\pi}{2} =$ $\quad$ $\operatorname{cosec} \frac{\pi}{2} =$

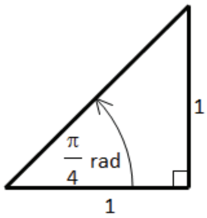
$\cos \frac{\pi}{2} =$ $\quad$ $\sec \frac{\pi}{2} =$

$\tan \frac{\pi}{2} =$ $\quad$ $\cot \frac{\pi}{2} =$

Fig 4.4.2

Let op dat die driehoeke hierbo nie regtig bestaan nie; ons gebruik ons verbeelding om te bepaal wat met die driehoek gebeur indien die sy en die skuinssy van die driehoek "op mekaar val"/

Note that the triangle above does not actually exist; we use our imagination in order to establish what happens with the triangle when one of its sides and the hypotenuse "co – incides".



$\sin\frac{\pi}{4} =$ $\quad$ $\operatorname{cosec}\frac{\pi}{4} =$

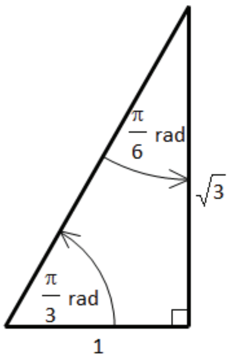
$\cos\frac{\pi}{4} =$ $\quad$ $\sec\frac{\pi}{4} =$

$\tan\frac{\pi}{4} =$ $\quad$ $\cot\frac{\pi}{4} =$

$\sin\frac{\pi}{3} =$ $\quad$ $\operatorname{cosec}\frac{\pi}{3} =$

$\cos\frac{\pi}{3} =$ $\quad$ $\sec\frac{\pi}{3} =$

$\tan\frac{\pi}{3} =$ $\quad$ $\cot\frac{\pi}{3} =$



$\sin\frac{\pi}{6} =$ $\quad$ $\operatorname{cosec}\frac{\pi}{6} =$

$\cos\frac{\pi}{6} =$ $\quad$ $\sec\frac{\pi}{6} =$

$\tan\frac{\pi}{6} =$ $\quad$ $\cot\frac{\pi}{6} =$

Fig. 4.4.3

Aangesien dit dikwels gebeur dat 'n hoek 'n waarde van meer as $\frac{\pi}{2}$ radiale besit, **moet ons trigonometrie kan doen met hoeke in al vier kwadrante van die platvlak**.

Because it often happens that an angle is larger than $\frac{\pi}{2}$ radians, ***we must be able to do trigonometry with angles in four quadrants of the flat plane***.

Voltooi onderstaande skets:

Complete the sketch below:

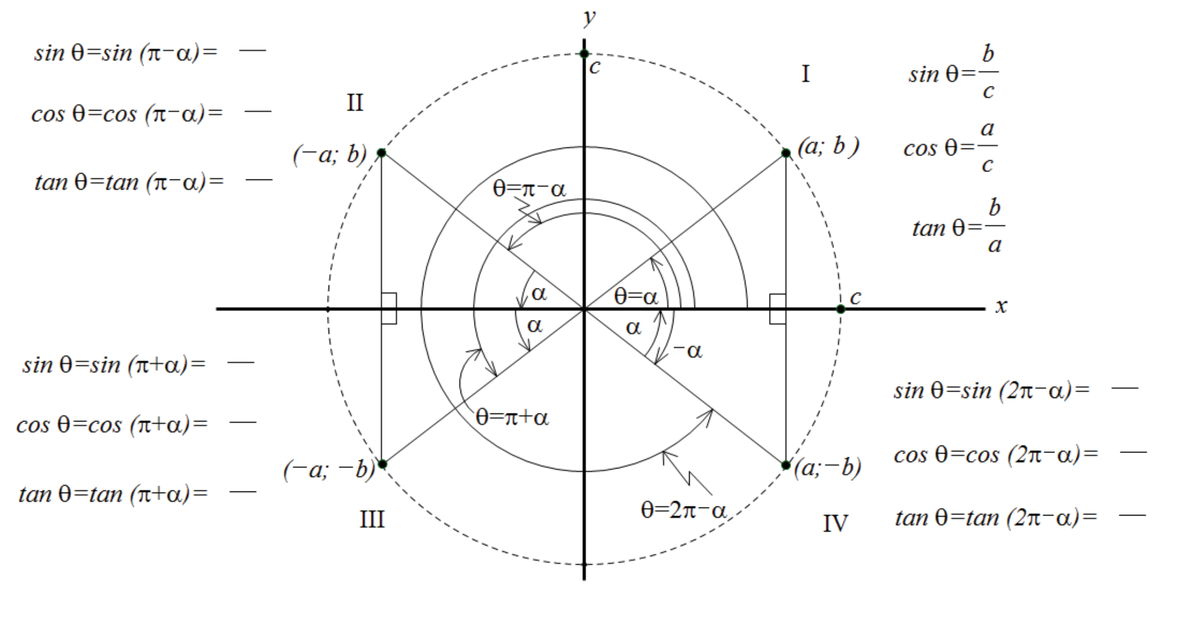
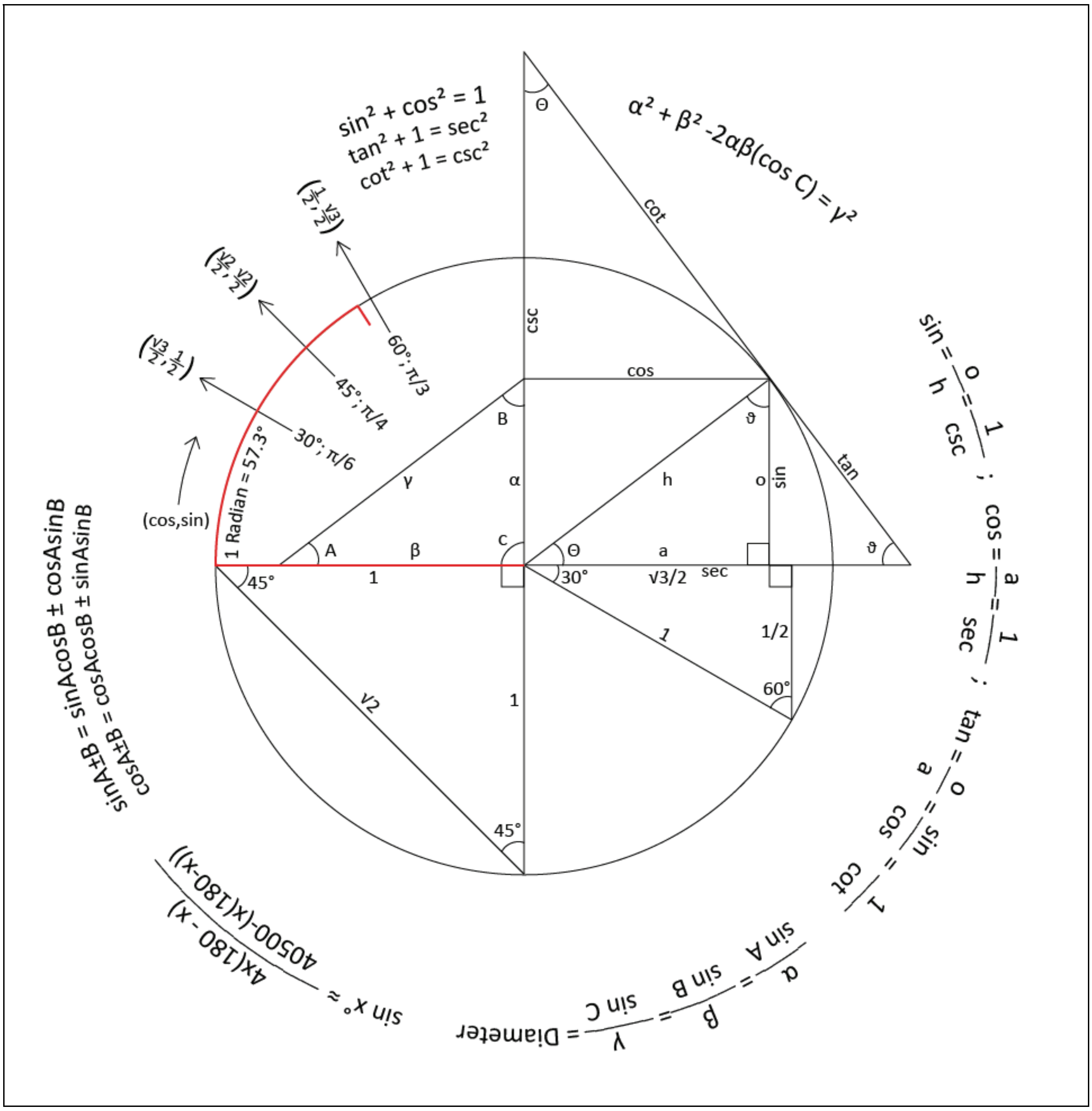


Fig. 4.4.4

Op die volgende bladsy is nog 'n interessante manier om die meeste van ons trigonometrie-kennis op te som:

On the next page you find another interesting way of summarizing most of our trigonometric knowledge:



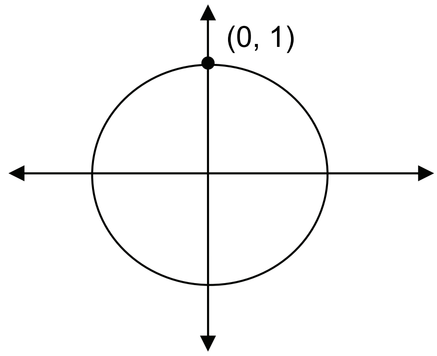
Meer oor spesiale hoeke

Gebruik die definisies in terme van die eenheidsirkel en bepaal die waardes van sinus, cosinus en tangens vir die volgende spesiale hoeke met behulp van die sketse:

More on special angles

*Use the definitions in terms of a unit circle and find the values of sine, cosine and tangent for the following **special angles** using the sketches supplied below:*

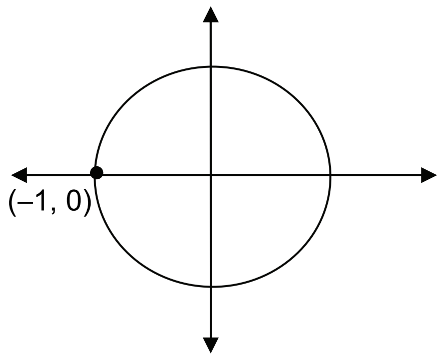
(a) $\cos\dfrac{\pi}{2}$ = ………………………………



(b) $\sin\dfrac{\pi}{2}$ = ………………………………

(c) $\tan\dfrac{\pi}{2}$ = ………………………………

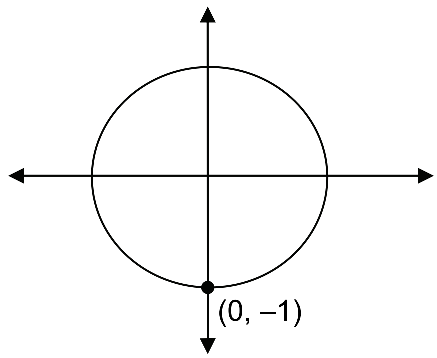
(d) $\cos\pi$ = ………………………………



(e) $\sin\pi$ = ………………………………

(f) $\tan\pi$ = ………………………………

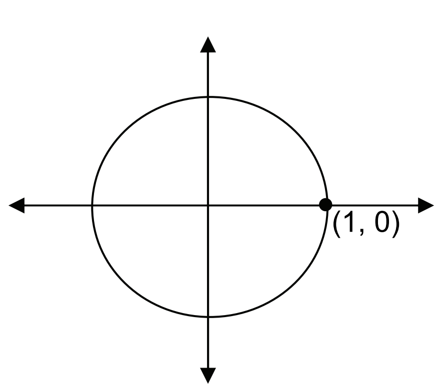
(g) $\cos\dfrac{3\pi}{2}$ = ………………………………



(h) $\sin\dfrac{3\pi}{2}$ = ………………………………

(i) $\tan\dfrac{3\pi}{2}$ = ………………………………

(j) $\cos 2\pi$ = ………………………………



(k) $\sin 2\pi$ = ………………………………

(l) $\tan 2\pi$ = ………………………………

Oefening 4.4

1. Bereken sonder 'n sakrekenaar die waarde van

Exercise 4.4

1. Calculate without a calculator the value of

1.1 $\sqrt{\tan^2\left(\frac{4}{3}\pi\right)+\sec^2\left(\frac{3}{4}\pi\right)\times 4\cos^2\left(\frac{11}{6}\pi\right)}$

1.2 $\left(\sin\left(\frac{2}{3}\pi\right)+\cos\left(\frac{5}{3}\pi\right)\right)^2+\operatorname{cosec}^2\left(\frac{\pi}{6}\right)$

1.3 $\sqrt{\cos\left(\frac{\pi}{2}\right)+\sin\left(\frac{5}{2}\pi\right)+3\cos(2\pi)}$

1.4 $3\cos^2(2\pi)+\frac{1}{3}\tan^2\left(\frac{4}{3}\pi\right)+2\sin\left(\frac{5}{6}\pi\right)+2\cos^5\left(\frac{3}{2}\pi\right)$

1.5 $\dfrac{\sin^2\left(\dfrac{5}{4}\pi\right)\tan\left(\dfrac{4}{3}\pi\right)\sec^2\left(\dfrac{11}{6}\pi\right)}{\tan\left(\dfrac{2}{3}\pi\right)}$

1.6 $\sin^2\left(\dfrac{4}{3}\pi\right)+\sin^2\left(\dfrac{7}{6}\pi\right)$

1.7 $\sec^2\left(\frac{\pi}{3}\right) - \tan^2\left(\frac{\pi}{3}\right)$

1.8 $\text{cosec}^2\left(\frac{2}{3}\pi\right) - \cot^2\left(\frac{2}{3}\pi\right)$

2. Gebruik die eerste skets in Fig. 4.4.1 en bewys dat

2. Use the first sketch in Fig. 4.4.1 and prove that

2.1 $\text{cosec}^2\theta = \cot^2\theta + 1$

2.2 $\sec^2\theta = \tan^2\theta + 1$

2.3 $\tan^4\theta + \tan^2\theta = \sec^4\theta - \sec^2\theta$

Wenk vir nr. 2/ *Hint for nr. 2*

Bewys dat/ *Prove that* $\sin^2\alpha+\cos^2\alpha=1$

Oplossing/ *Solution*

Linkerkant/$LHS=\sin^2\alpha+\cos^2\alpha$

$=\left(\frac{a}{c}\right)^2+\left(\frac{b}{c}\right)^2$ Volgens 2e skets in Fig. 4.4.1/ Using 2nd sketch in Fig. 4.4.1

$=\frac{a^2}{c^2}+\frac{b^2}{c^2}$

$=\frac{a^2+b^2}{c^2}$

$=\frac{c^2}{c^2}$ Volgens Pythagoras is/ According to Pythagoras is $a^2+b^2=c^2$

$=1$

Regterkant/$RHS=1$

$\therefore$ Linkerkant/$LHS=$ Regterkant/RHS dus / *so* $\sin^2\alpha+\cos^2\alpha=1$

3. Vereenvoudig na eenvoudigste vorm:

3. Simplify to simplest form:

3.1 $$\frac{\sin(\pi-\theta)\cot(\pi+\theta)\sec(2\pi-\theta)}{\cos^2(\pi+\theta)+\cos^2\left(\frac{\pi}{2}-\theta\right)}$$

3.2 $$\frac{\tan(\pi-\theta)\sqrt{1-\sin^2\theta}}{\cos\left(\frac{\pi}{2}-\theta\right)}$$

3.3 $\dfrac{\sin(\pi + A)\cos(2\pi - A)}{\cos\left(\dfrac{\pi}{2} - A\right)\cos(2\pi + A)}$

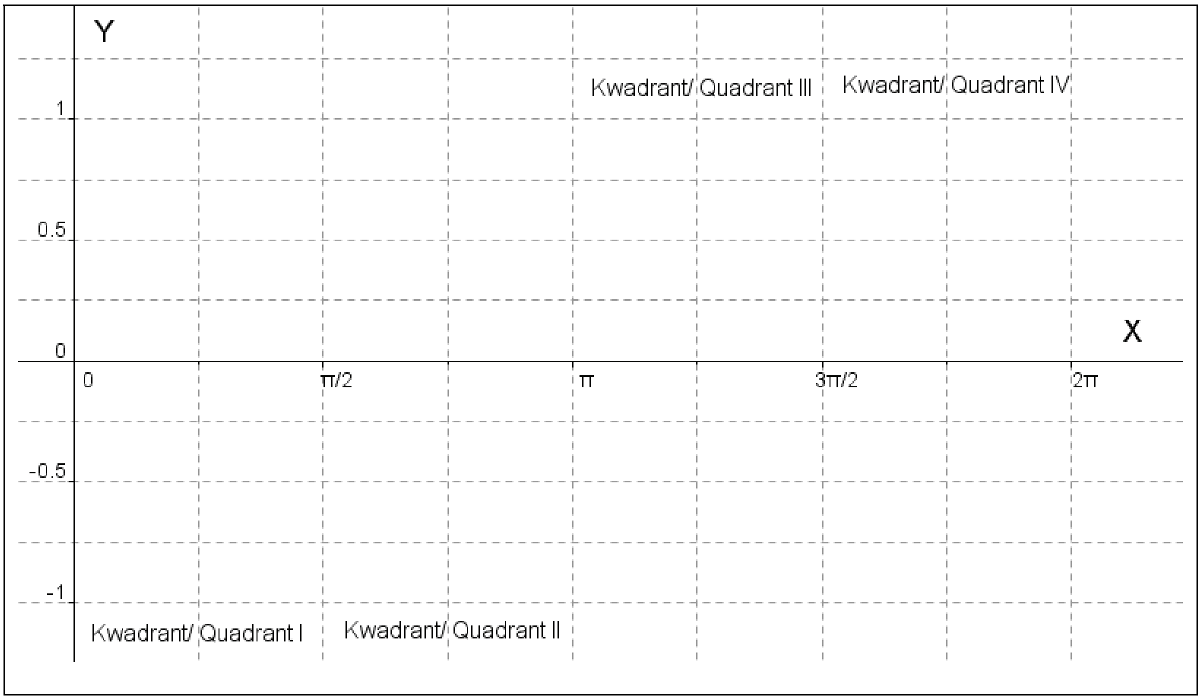
4. Voltooi die volgende drie voorstellings van die basiese trigonometriese grafieke:

4. Complete the following three representatios of the basic trigonometric graphs:

4.1 $y = \sin x$ vir/ *for* $x \in [0; 2\pi]$

Wenk/*Hint:*

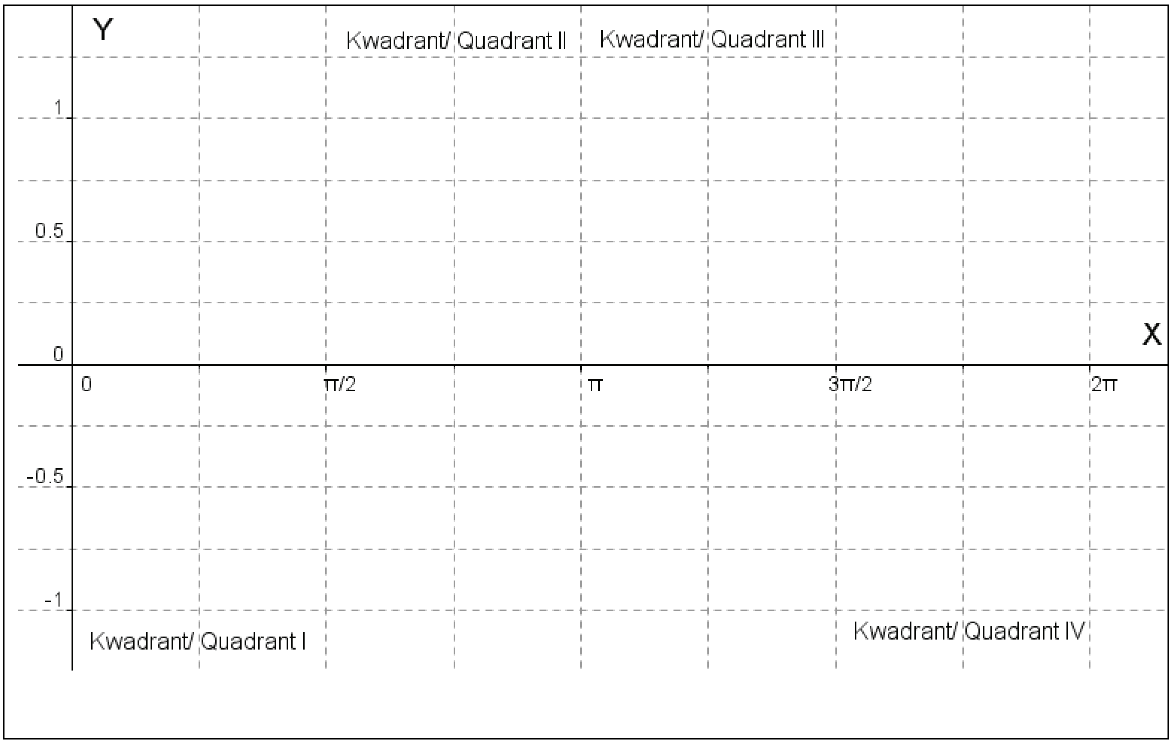
x	0	$\frac{1}{4}\pi$	$\frac{1}{2}\pi$	$\frac{3}{4}\pi$	π	$\frac{5}{4}\pi$	$\frac{3}{2}\pi$	$\frac{7}{4}\pi$	2π
y									



4.2 $y = \cos x$ vir/ *for* $x \in [0; 2\pi]$

Wenk/*Hint:*

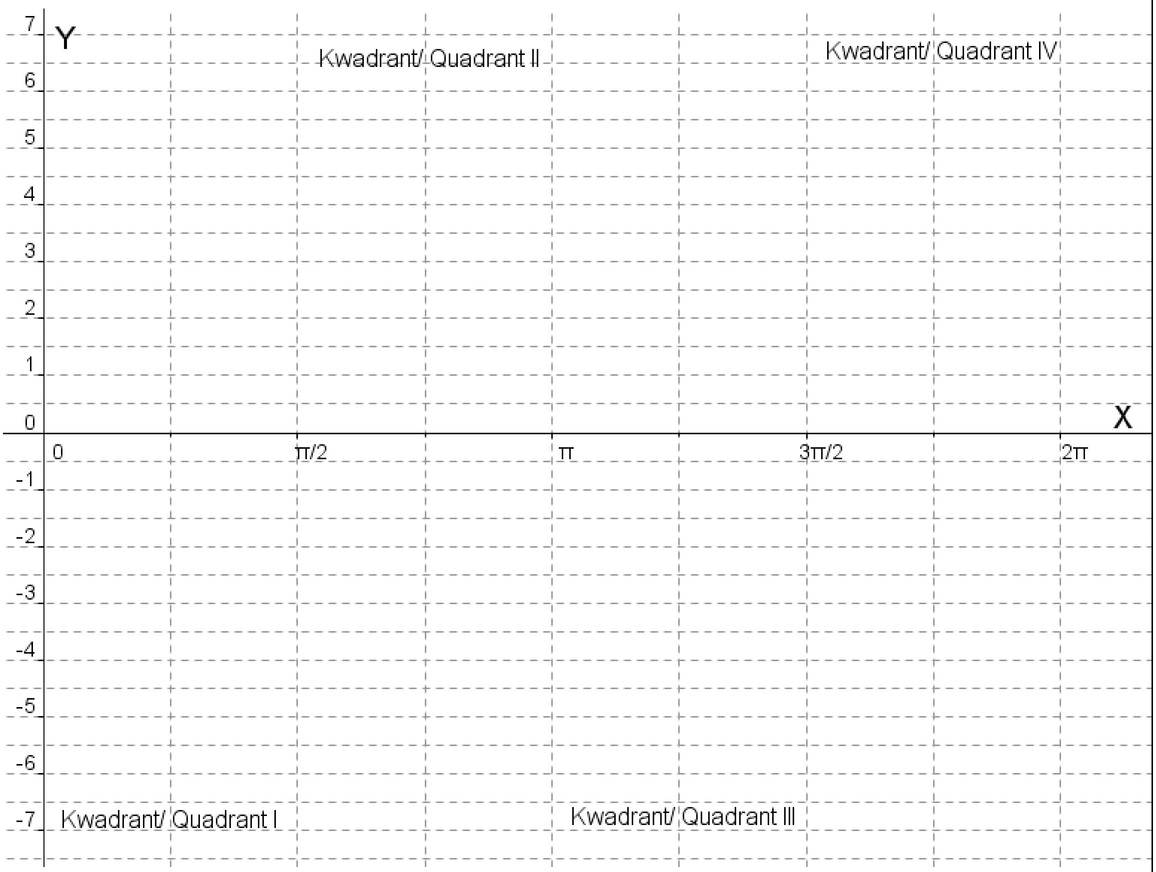
x	0	$\frac{1}{4}\pi$	$\frac{1}{2}\pi$	$\frac{3}{4}\pi$	π	$\frac{5}{4}\pi$	$\frac{3}{2}\pi$	$\frac{7}{4}\pi$	2π
y									



4.3 $y = \tan x$ vir/ *for* $x \in [0; 2\pi]$

Wenk/*Hint:*

x	0	$\frac{\pi}{4}$	$\frac{45\pi}{100}$	$\frac{\pi}{2}$	$\frac{55\pi}{100}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{145\pi}{100}$	$\frac{3\pi}{2}$	$\frac{155\pi}{100}$	$\frac{7\pi}{4}$	2π
y													



5. Los die volgende trigonometriese vergelykings op vir waardes van θ sodat $0 \leq \theta \leq 2\pi$:

5. Solve the following trigonometric equations for values of θ such that $0 \leq \theta \leq 2\pi$:

5.1 $\sin\theta = -\frac{1}{2}$

5.2 $2\cos\theta = \sqrt{3}$

5.3 $\sqrt{3}\tan\theta + 1 = 0$

5.4 $\sec 2\theta + 2 = 0$

5.5 $\sqrt{3}\text{cosec}2\theta + 2 = 0$

NWU®

6. Los die volgende trigonometriese vergelykings op vir alle radiaal-waardes van θ, met ander woorde, vind die algemene oplossings:

6. Solve the following trigonometric equations for all radian values of θ, in other words – find the general solutions:

6.1 $\cos(x+\frac{\pi}{3})=\sin(3x-\frac{\pi}{3})$

6.2 $\sin(\cos x)=0$

4.5 Die som- en verskil-formules/ *The sum and difference formulae*

Uit die definisies van die trigonometriese funksies kan die volgende belangrike indentiteite afgelei word; in hierdie kursus het ons nie tyd om hul afleidings te bespreek nie, maar u kan dit gerus self naslaan:

From the definitions of the trigonometric functions the following important identities may be derived; in this course we do not have time to discuss their derivations but you may research that on your own:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

Oefening 4.5

1. Indien gegee is dat $\sin A = \frac{5}{13}$ met $0 \le A \le \frac{\pi}{2}$ en $\cos B = -\frac{3}{5}$ met $\pi \le B \le \frac{3\pi}{2}$, bereken sonder 'n sakrekenaar:

Exercise 4.5

1. If it is given that $\sin A = \frac{5}{13}$ with $0 \le A \le \frac{\pi}{2}$ and $\cos B = -\frac{3}{5}$ with $\pi \le B \le \frac{3\pi}{2}$, calculate without a calculator:

1.1 $\sin(A - B)$

1.2 $\cos(A+B)$

1.3 $\tan(B-A)$

2. Bereken sonder 'n sakrekenaar die waarde van $\cos\left(\frac{\pi}{12}\right)$ deur gebruik te maak van $\cos\left(\frac{\pi}{4}-\frac{\pi}{6}\right)$:

2. Calculate without a calculator the value of $\cos\left(\frac{\pi}{12}\right)$ *by making use of* $\cos\left(\frac{\pi}{4}-\frac{\pi}{6}\right)$*:*

3. Bewys dat $\tan(\pi+\theta)=\tan\theta$

3. Bewys dat $\tan(\pi+\theta)=\tan\theta$

4. Bereken die sonder 'n sakrekenaar die waarde van $\cos 23^\circ \cos 67^\circ - \sin 23^\circ \sin 67^\circ$.

4. Calculate without a calculator the value of $\cos 23^\circ \cos 67^\circ - \sin 23^\circ \sin 67^\circ$.

5. Bereken sonder 'n sakrekenaar die waarde van $\frac{\tan 18^\circ + \tan 27^\circ}{1 - \tan 18^\circ \tan 27^\circ}$

5. *Calculate without a calculator the value of* $\frac{\tan 18^\circ + \tan 27^\circ}{1 - \tan 18^\circ \tan 27^\circ}$.

4.6 Die dubbelhoekformules/ *The double angle formulae*

Uit die som- en verskilformules kan die volgende belangrike indentiteite afgelei word:

From the sum and difference formulae the following important identities may be derived:

$$\begin{aligned}\sin 2A &= 2\sin A\cos A\\ \cos 2A &= \cos^2 A - \sin^2 A\\ &= 2\cos^2 A - 1\\ &= 1-2\sin^2 A\\ \tan 2A &= \frac{2\tan A}{1-\tan^2 A}\end{aligned}$$

Oefening 4.6 **Exercise 4.6**

1. Bewys dat/ *Prove that* $\sin 2A = 2\sin A\cos A$ deur soos volg te begin/ *by starting as follows*:

$$\begin{aligned}\text{Linkerkant/ } LHS &= \sin 2A\\ &= \sin(A+A)\end{aligned}$$

2. Bewys dat/ *Prove that* $\cos 2A = 1-2\sin^2 A$.

3. Gebruik u resultaat uit vraag 2 om 'n indentiteit vir $\sin^2 A$ af te lei in terme van $\cos 2A$ /

   Use your result from question 2 in order to derive an identity for $\sin^2 A$ *in terms of* $\cos 2A$

4. Bewys dat/ *Prove that* $\cos 2A = 2\cos^2 A - 1$

5. Gebruik u resultaat uit vraag 4 om 'n indentiteit vir $\cos^2 A$ af te lei in terme van $\cos 2A$ /

   Use your result from question 4 in order to derive an identity for $\cos^2 A$ *in terms of* $\cos 2A$

6. Bewys die indentiteit/ *Prove the identity*: $\dfrac{\sin 2\phi}{1+\cos 2\phi} = \tan\phi$

7. Bewys die indentiteit/ *Prove the identity*: $\sin(x+y)\cdot\sin(x-y) = \sin^2 x - \sin^2 y$

8. Bewys die indentiteit/ *Prove the identity*: $\cos\frac{28\pi}{6} = \cos\frac{\pi}{6}$

9. Bewys die indentiteit/ *Prove the identity*: $\cos\frac{28\pi}{3} = \cos(-\frac{2\pi}{3})$

10. Bewys die indentiteit/ *Prove the identity*: $\dfrac{\sec x - \cos x}{\sin x + \tan x} = \operatorname{cosec} x - \cot x$

Blaai na die bylae van hierdie boek en voltooi Werkkaart 4.

Turn to the addendum of this book and complete Worksheet 4.

5 Absolute waardes en limiete/ *Absolute values and limits*

Leerdoelstellings vir hierdie leereenheid

Na afhandeling van hierdie leereenheid moet die student in staat wees om die volgende te doen:

1. **Ongelykhede op te los**
2. **Die absolute waarde-bewerking op algebraïese uitdrukkings toe te pas**
3. **Die absolute waarde-funksie as 'n stuksgewyse funksie te definieer**
4. **Die absolute waarde-funksie as die "afstand"-funksie te interpreteer en grafies voor te stel**
5. **Die limiet van 'n funksie in terme van linkerlimiete en regterlimiete in die omgewing van 'n punt te bepaal**
6. **Uitspraak te kan lewer oor die kontinuïteit van 'n funksie in 'n bepaalde punt**
7. **Sekere limiete te bereken**
8. **Die epsilon-delta-definisie van die limiet van 'n funksie toe te pas om te bewys dat die limiet van 'n funksie in 'n bepaalde punt bestaan**

Learning aims for this study unit

Upon completion of this study unit the student must be able to do the following:

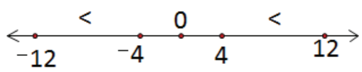
1. ***Solve inequalities***
2. ***Apply the absolute value operation to algebraic expressions***
3. ***Define the absolute value function as a piece-wise function***
4. ***Interpret the absolute value function as the "distance"-function and to represent it graphically***
5. ***Determine the limit of a function in terms of left-sided limits and right-sided limits in the vicinity of a point***
6. ***Give judgment regarding the continuity of a function in a particular point***
7. ***Compute certain limits***
8. ***Apply the epsilon-delta definition of the limit of a function in order to prove that the limit of a function in a particular point exists***

5.1 Ongelykhede/ *Inequalities*

Indien ons die $=$ in 'n vergelyking vervang met een van die relasietekens $<$, $\leq$, $>$ of $\geq$ dan verkry ons 'n ongelykheid.

Sulke ongelykhede kan op soortgelyke wyse opgelos word as gewone vergelykings, solank die volgende in gedagte gehou word:

From the sum and difference formulae the following important identities may be derived:

Eienskap/ *Property*	Voorbeeld/ *Example*
As/ *if* $a<b$ dan is/ *then* $-a>-b$ (vermenigvuldiging of deling met -1 verander die ongelykheidsteken/ *Multiplication or division by* -1 *changes the sign of the relationship symbol*)	$4<12$ is waar/ *is true* Maar/ *but* $-4<-12$ is onwaar/ *is false* Dus/ *So* $-4>-12$ of/ *or* $-12<-4$ 
As/ *if* $a<b$ dan is/ *then* $a+c<b+c$ en/ *and* $a-c<b-c$	As/ *if* $x-3<0$ dan is/ *then* $x-3+3<0+3$ en dus is / and consequently $x<3$
As/ *if* $a<b$ en/ *and* $c>0$ dan is/ *then* $ac<bc$ en/ *and* $\frac{a}{c}<\frac{b}{c}$	As/ *if* $\frac{1}{2}x<4$ dan is/ *then* $(2)\frac{1}{2}x<(2)4$ sodat/ *which implies* $x<8$
As/ *if* $a<b$ en/ *and* $c<0$ dan is/ *then* $ac>bc$ en/ *and* $\frac{a}{c}>\frac{b}{c}$	As/ *if* $-\frac{x}{5}<6$ dan is/ *then* $(-5)\left(-\frac{x}{5}\right)>(-5)(6)$ sodat/ *which implies* $x>-30$
As/ *if* $a<b$ en/ *and* $b<c$ dan is/ *then* $a<c$	As/ *if* $a<x$ en/ *and* $x<2$ dan is/ *then* $a<2$

Die dodelikste gevaar by die oplos van ongelykhede is waarskynlik by rasionale ongelykhede, wat NIE soos rasionale vergelykings gehanteer kan word NIE.

Beskou die volgende voorbeelde aandagtig:

The most lethal danger associated with the solution of inequalities probably occurs in the case of rational inequalities, which CANNOT be treated the same way as rational equations

Study the following examples with great concentration:

Rasionale vergelyking/ *Rational equation*

$$\frac{x-3}{x+2}=2$$

$$\therefore \frac{x-3}{x+2}\times\frac{x+2}{1}=2\times\frac{x+2}{1}$$

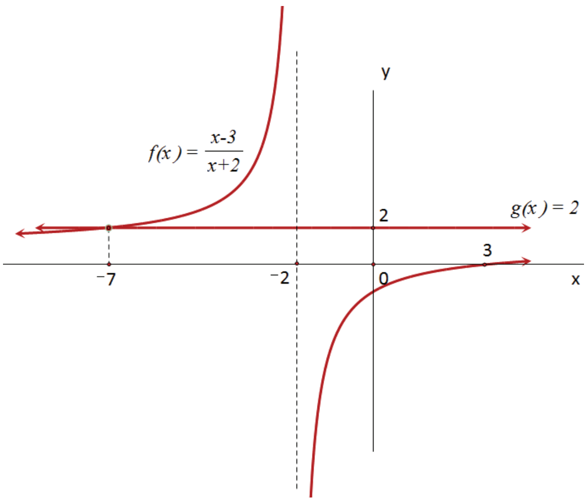
$$\therefore x-3=2x+4$$

$$\therefore -x=7$$

$$\therefore x=-7$$

DIE KGV WORD GEBRUIK OM WEG TE DOEN MET DIE BREUKVORM/ *THE LCM IS USED IN ORDER TO DO AWAY WITH THE FRACTION FORM*

Grafies/ *Graphically:*



Rasionale ongelykheid/ *Rational inequality*

$$\frac{x-3}{x+2}>2$$

$$\therefore \frac{x-3}{x+2}-2>0$$

$$\therefore \frac{x-3}{x+2}-2\times\frac{x+2}{x+2}>0$$

$$\therefore \frac{x-3}{x+2}-\frac{2x-4}{x+2}>0$$

$$\therefore \frac{x-3-2x-4}{x+2}>0$$ BEHOU BREUKVORM/

KEEP FRACTION FORM!

$$\therefore \frac{-x-7}{x+2}>0$$

$\therefore \frac{x+7}{x+2}<0$ so/so $\frac{x+7}{x+2}$ is negatief/negative

Ons stel nou intervalle saam uit punte/ *We now set up intervals from the points*

$x=-7$ en/ *and* $x=-2$:

interval	$(-\infty;-7)$ of/*or* $x<-7$	$(-7;-2)$ of/*or* $-7<x<-2$	$(-2;\infty)$ of/*or* $x>-2$
monsterpunt/ *sample point*	-8	-5	-1
waarde van/ *value of* $\frac{x+7}{x+2}$	0,167	−0,667	6

$\frac{x+7}{x+2}$ is negatief op/ *is negative on*

$-7<x<-2$.

Die oplossing van/ *The solution of* $\frac{x-3}{x+2}>2$ is dus/ *is therefore* $-7<x<-2$, ook geskryf as/ *also written as* $(-7;-2)$

(vergelyk met grafiek links/ *compare with the graph above left*)

Rasionale vergelyking/ *Rational equation*

$$\frac{2x+1}{x-1}-\frac{2}{x-3}=1$$

$$\therefore \frac{(x-1)(x-3)}{1}\times\left(\frac{2x+1}{x-1}-\frac{2}{x-3}\right)=1\times\frac{(x-1)(x-3)}{1}$$

$$\therefore (2x+1)(x-3)-2(x-1)=(x-1)(x-3)$$

$$\therefore 2x^2-7x-1=x^2-4x+3$$

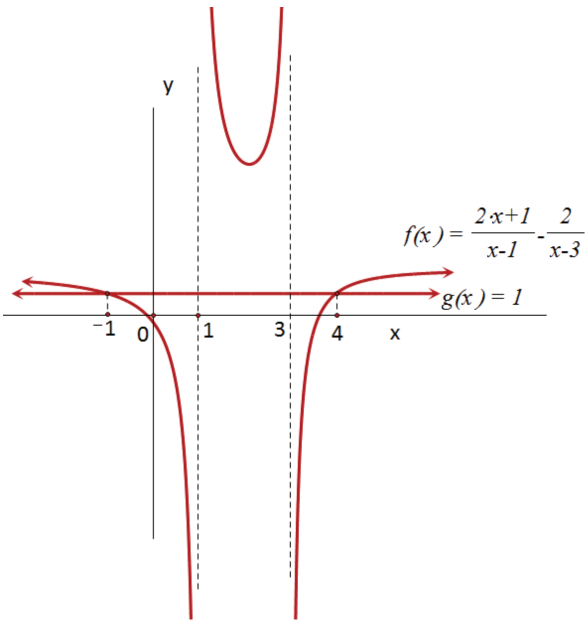
$$\therefore x^2-3x-4=0$$

$$\therefore (x-4)(x+1)=0$$

$$\therefore x=4 \text{ of / } or \quad x=-1$$

DIE KGV WORD GEBRUIK OM WEG TE DOEN MET DIE BREUKVORM/ *THE LCM IS USED IN ORDER TO DO AWAY WITH THE FRACTION FORM*

Grafies/ *Graphically:*



Rasionale ongelykheid/ *Rational inequality*

$$\frac{2x+1}{x-1}-\frac{2}{x-3}<1$$

$$\therefore \frac{2x+1}{x-1}-\frac{2}{x-3}-1<0$$

$$\therefore \frac{(2x+1)(x-3)}{(x-1)(x-3)}-\frac{2(x-1)}{(x-3)(x-1)}-\frac{(x-3)(x-1)}{(x-3)(x-1)}<0$$

$$\therefore \frac{x^2-3x-4}{(x-1)(x-3)}<0$$

$$\therefore \frac{(x-4)(x+1)}{(x-1)(x-3)}<0$$

Ons stel nou intervalle saam uit punte/ *We now set up intervals from the points*

$x=4$, $x=-1$, $x=1$ en/ *and* $x=3$:

interval	$-\infty<x<-1$ of/or $(-\infty;-1)$	$-1<x<1$ of/or $(-1;1)$	$1<x<3$ of/or $(1;3)$	$3<x<4$ of/or $(3;4)$	$x>4$ of/or $(4;\infty)$
monsterpunt/ *sample point*	-2	0	2	3,5	5
waarde van/ *value of* $\frac{(x-4)(x+1)}{(x-1)(x-3)}$	0,4	$-1{,}333$	6	$-1{,}8$	0,75

$\frac{(x-4)(x+1)}{(x-1)(x-3)}$ is negatief op/ *is negative on*

$-1<x<1$ en/*and* $3<x<4$.

Die oplossing is dus/ *The solution is therefore* $(-1;1)\cup(3;4)$

(vergelyk met grafiek links/ *compare with the graph left*)

Oefening 5.1 | ***Exercise 5.1***

Los op die volgende ongelykhede: | *Solve the following inequalities:*

1. $\dfrac{x-3}{x+1} \geq 0$

2. $\dfrac{2x+1}{x-5} \leq 3$

3. $\frac{1+x}{1-x}-\frac{1-x}{1+x}\leq -1$

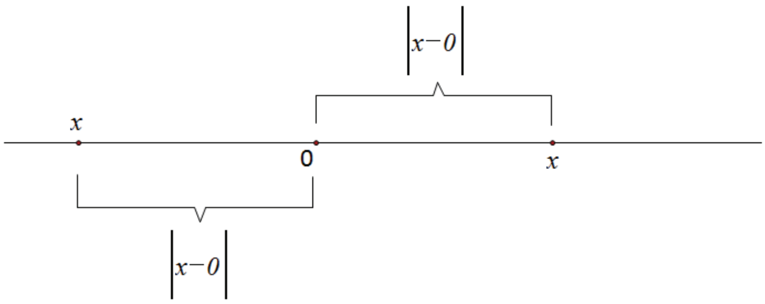
5.2 Absolute waardes/ *Absolute values*

Die afstand-definisie

$|x|$ beteken $|x-0|$ en dit beteken die grootte van die afstand tussen die punt 0 op die getallelyn en die punt x.

The distance definition

$|x|$ means $|x-0|$ and that means the distance between 0 on the number line and the point x.



Dit is duidelik dat x aan enige kant van die punt 0 kan lê; die afstand $|x-0|$ is altyd positief – so, die waarde van $|x|$ vir enige waarde van x sal altyd positief wees.

It is clear that x may be located on either side of 0; the distance $|x-0|$ is always positive – so, the value of $|x|$ for any value of x will always be positive.

Die getal $|x-0|$ kan verkry word deur 'n passer se radius op $|x|$ te stel en die punt op die nulpunt neer te sit.

The number Die getal $|x-0|$ may be obtained by setting the radius of a compass on $|x|$ and putting the sharp pin in the zero point.

Dan kan die punt x aan weerskante van die nulpunt afgemerk word.

Then the point x may be marked off on both sides of the zero point.

Die formele algebraïese definisie

The formal algebraic definition

$$|x| = \begin{cases} x & \text{as/}if\ \ x \geq 0 \\ -x & \text{as/}if\ \ x < 0 \end{cases}$$ of in die algemeen/*or in general*

$$|x-a| = |(x-a)-0| = \begin{cases} x-a & \text{as/}if\ \ x-a \geq 0 \\ -(x-a) & \text{as/if}\ \ x-a < 0 \end{cases}$$

Die formule algebraïese definisie gee ons 'n manier om die absolute waarde-funksie, gedefinieer deur $f(x) = a|bx - d| + h$ maklik te skets.

Die geheim is om die funksie as 'n stuksgewyse funksie te hanteer; dan gedra die absolute waarde-funksie haarself soos 'n kombinasie van twee beperkte reguit lyn-grafieke.

Die knakpunt is die punt op die grafiek waaromheen die grafiek simmetries is:

The formal algebraic definition provides us with a way to easily sketch the absolute value function, defined by $f(x) = a|bx - d| + h$ *.*

The secret is to treat the function as a piece-wise defined function; then the absolute value function behaves like a combination of two constrained straight line graphs.

The vertex is the point on the graph about which the graph is symmetrical:

$$
\begin{aligned}
f(x) &= a|bx - d| + h \\
\therefore f(x) &= \begin{cases} a(bx - d) + h & \text{as/if} \quad bx - d \geq 0 \\ -a(bx - d) + h & \text{as/if} \quad bx - d < 0 \end{cases} \\
&= \begin{cases} abx - ad + h & \text{as/if} \quad bx \geq d \\ -abx + ad + h & \text{as/if} \quad bx < d \end{cases} \\
&= \begin{cases} \overbrace{(ab)}^{m}x + \overbrace{(-ad + h)}^{c} & \text{as/if} \quad x \geq \dfrac{d}{b} \\ \left(\underbrace{-ab}_{m}\right)x + \underbrace{(ac + h)}_{c} & \text{as/if} \quad x < \dfrac{d}{b} \end{cases}
\end{aligned}
$$

Die knakpunt is dus by/ So the vertex is at $x = \dfrac{d}{b}$

Vervang/ *substitute* $x = \dfrac{d}{b}$ in/ *into* $y = \overbrace{(ab)}^{m}x + \overbrace{(-ad + h)}^{c}$ om die/ *in order to obtain the* y-ko-oordinaat/ *co-ordinate* te verkry.

Voorbeeld/ *Example*

Skets die grafieke van die volgende funksies/ *Sketch graphs of the following functions:*

1. $y = 2\left|\dfrac{1}{4}x - 2\right| + 4$

2. $y = -\dfrac{1}{4}|3x + 5| - 3$

Oplossing/ *Solution*

1. $$y = 2\left|\frac{1}{4}x - 2\right| + 4$$

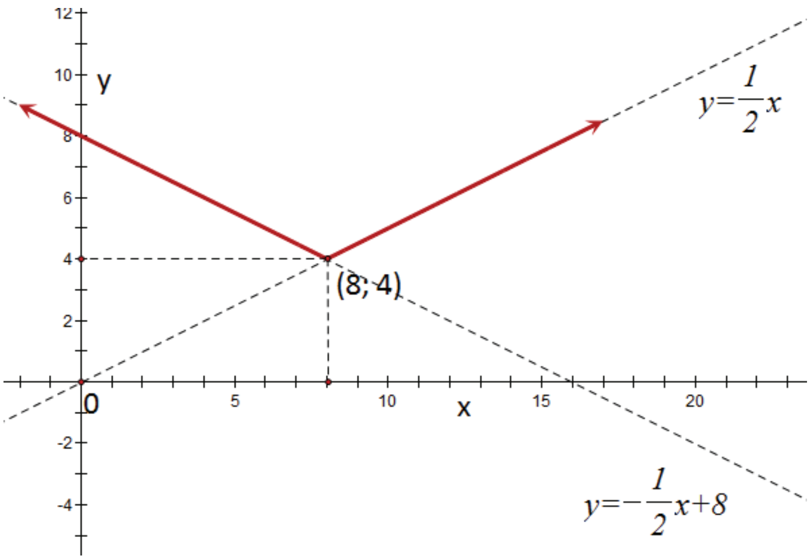
$$\therefore y = \begin{cases} 2\left(\frac{1}{4}x - 2\right) + 4 & \text{as/ if} \quad \frac{1}{4}x - 2 \geq 0 \\ -2\left(\frac{1}{4}x - 2\right) + 4 & \text{as/ if} \quad \frac{1}{4}x - 2 < 0 \end{cases}$$

$$= \begin{cases} \frac{1}{2}x - 4 + 4 & \text{as/ if} \quad \frac{1}{4}x \geq 2 \\ -\frac{1}{2}x + 4 + 4 & \text{as/ if} \quad \frac{1}{4}x < 2 \end{cases}$$

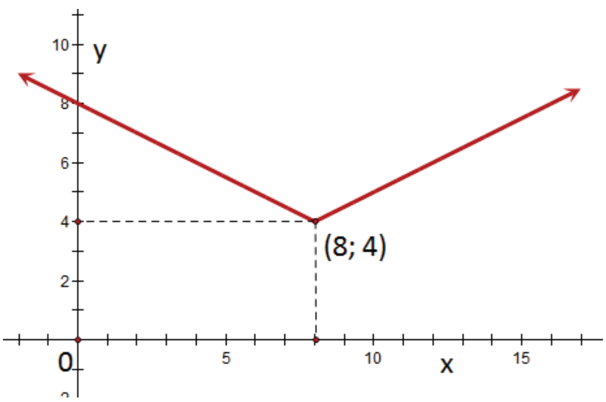
$$= \begin{cases} \frac{1}{2}x & \text{as/ if} \quad x \geq 8 \\ -\frac{1}{2}x + 8 & \text{as/ if} \quad x < 8 \end{cases}$$

Knakpunt/ *vertex*: $(8; 4)$

Dit lewer/ *this yields*:



Normaalweg wys ons nie die twee "hulplyne" nie/ *Normally, we do not show the two "auxilliary lines":*



2. $y=-\frac{1}{4}|3x+5|-3$

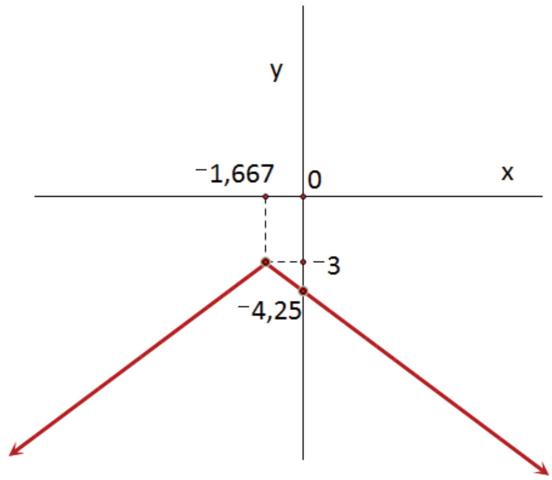
$$\therefore y=\begin{cases}-\frac{1}{4}(3x+5)-3 & \text{as/ if} \quad 3x+5\geq 0\\ +\frac{1}{4}(3x+5)-3 & \text{as/ if} \quad 3x+5<0\end{cases}$$

$$=\begin{cases}-\frac{3}{4}x-\frac{5}{4}-3 & \text{as/ if} \quad 3x\geq -5\\ \frac{3}{4}x+\frac{5}{4}-3 & \text{as/ if} \quad 3x<-5\end{cases}$$

$$=\begin{cases}-\frac{3}{4}x-\frac{17}{4} & \text{as/ if} \quad x\geq -\frac{5}{3}\\ \frac{3}{4}x-\frac{7}{4} & \text{as/ if} \quad x<-\frac{5}{3}\end{cases}$$

Knakpunt/ *vertex*: $\left(-\frac{5}{3};-3\right)$

Dit lewer/ *this yields*:



Meetkundige definisie

Hoe kan ons 'n passer en 'n getallelyn gebruik om betekenis te gee aan iets soos $|x-a|=r$? (Wat is die meetkundige betekenis van $|x-a|=r$?)

$|x-a|=r$ beteken $|(x-a)-0|=r$ en dit beteken $x-a$ lê presies r eenhede vanaf 0 op die getallelyn, so plaas die passer se punt op 0 en stel die radius op r. Trek dan 'n sirkel om 0 en let op waar die sirkel die getallelyn sny. $x-a$ lê op die snypunte

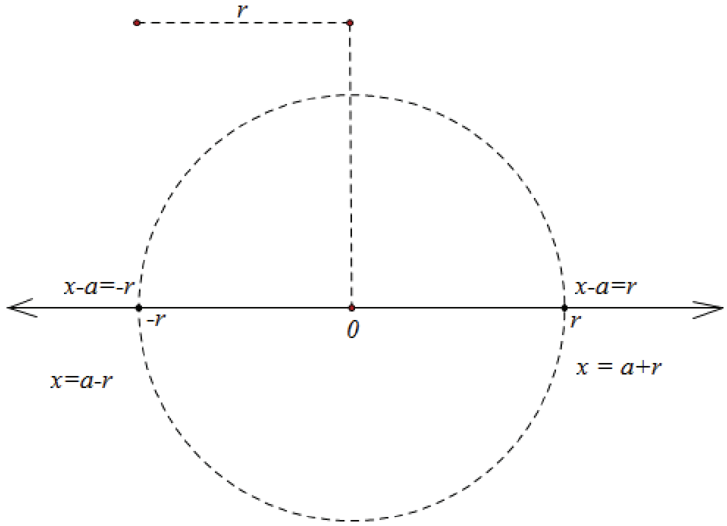
Geometric definition

How may we use a compass and a number line to assign meaning to something like $|x-a|=r$? (What is the geometric meaning of $|x-a|=r$?)

$|x-a|=r$ *means* $|(x-a)-0|=r$ *and that means that* $x-a$ *lies precisely* r *units from* 0 *on the number line, so place the pin of the compass on* 0 *and set the radius on* r *. Construct a circle around* 0 *and note where the circle intersects the number line.* $x-a$

van die sirkel met die lyn, dit is die punte $-r$ en $-r$.

lies on the points of intersection of the circle with the line, that is the points $-r$ and $-r$.



Meetkundige definisie/ *Geometric definition*:

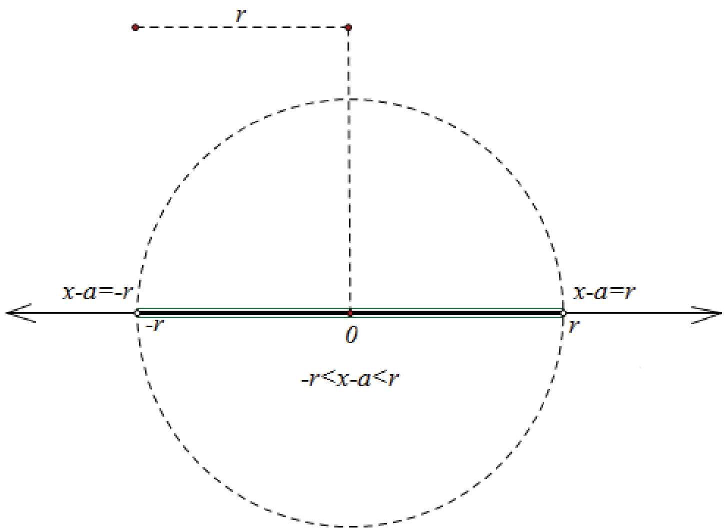
$|x-a|=r$ impliseer dat/ *implies that* $x=a-r$ of $x=a+r$

Hoe kan ons 'n passer en 'n getallelyn gebruik om betekenis te gee aan iets soos $|x-a|<r$ of selfs $|x-a|>r$? (Wat is die meetkundige betekenis van $|x-a|<r$ en $|x-a|>r$?)

How may we use a compass and a number line in order to assign meaning to something like $|x-a|<r$ or even $|x-a|>r$? (What is the geometric meaning of $|x-a|<r$ and $|x-a|>r$?)

$|x-a|<r$ beteken $|(x-a)-0|<r$ en dit beteken $x-a$ lê minder as r eenhede vanaf 0 op die getallelyn, so plaas die passer se punt op 0 en stel die radius op r. Trek dan 'n sirkel om 0 en let op waar die sirkel die getallelyn sny. $x-a$ lê enige plek op die deel van die getallelyn tussen die snypunte van die sirkel met die lyn.

$|x-a|<r$ means $|(x-a)-0|<r$ and this indicates that $x-a$ lies less than r units from 0 on the number line, so place the pin of the compass on 0 and set the radius on r. Next, construct a circle around 0 and note where the circle intersects the number line. $x-a$ lies anywhere on the part of the number line between the points of intersection of the circle with the line.



$|x-a|<r$, dit is/ *that is* $|(x-a)-0|<r$ impliseer dat/ *implies that* $-r<x-a<r$

waaruit volg/ *from which follows that* $a-r<x<a+r$

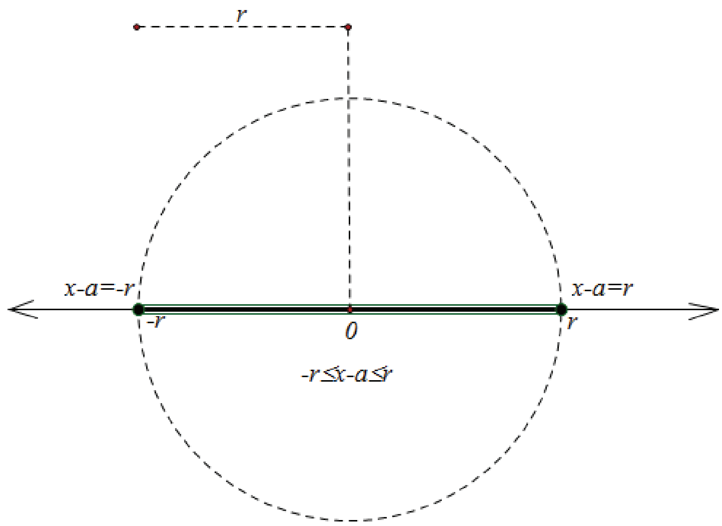
ook geskryf as/ *also written as*: $(a-r;a+r)$

Vir die geval $|x-a|\leq r$ oftewel $|(x-a)-0|\leq r$ lyk die situasie soos volg:

For the case $|x-a|\leq r$ or just as well $|(x-a)-0|\leq r$ the situation is as follows:

$$|x-a| \le r$$

$$|(x-a)-0| \le r$$



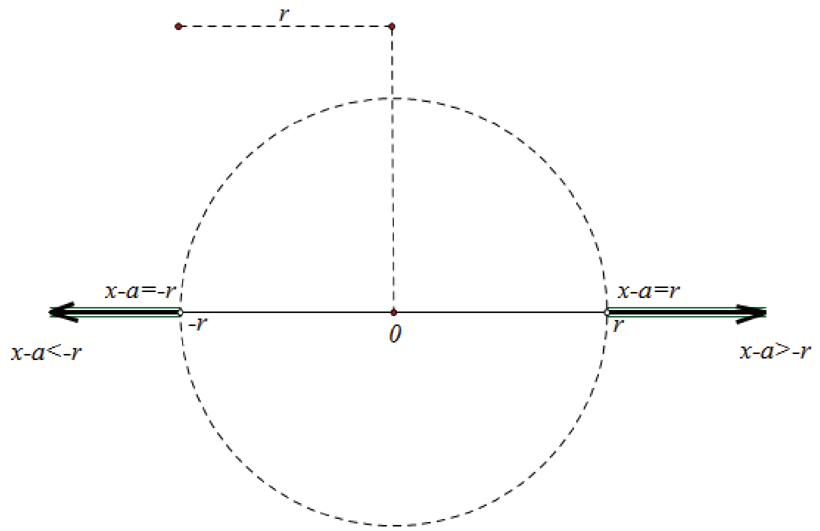
$|x-a| \le r$, dit is/ *that is* $|(x-a)-0| \le r$ impliseer dat/ *implies that* $-r \le x-a \le r$

waaruit volg/ *from which follows that* $a-r \le x \le a+r$

ook geskryf as/ *also written as*: $[a-r; a+r]$

Net so: $|x-a| > r$ beteken $|(x-a)-0| > r$ en dit beteken $x-a$ lê meer as r eenhede vanaf 0 op die getallelyn, so $x-a$ lê enige plek op die deel van die getallelyn links en regs van die snypunte van die sirkel met die lyn.

Similarly: $|x-a| > r$ *means* $|(x-a)-0| > r$ *and that means* $x-a$ *lies more than* r *units from* 0 *on the number line,* $x-a$ *lies anywhere on the section of the number line to the left or to the right of the points of intersection of the circle with the line.*



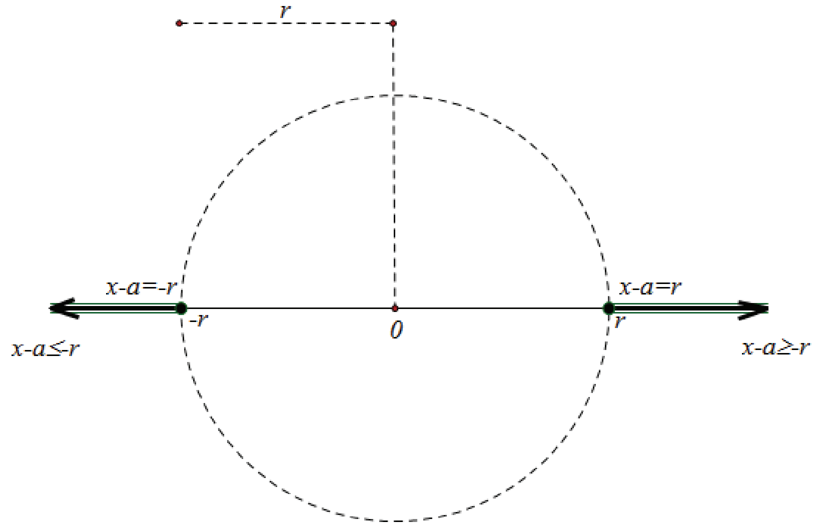
$|x-a|>r$, dit is/ *that is* $|(x-a)-0|>r$ impliseer dat/ *implies that* $x-a<-r$ of/or $x-a>r$

waaruit volg/ *from which follows that* $x<a-r$ of/or $x>a+r$

ook geskryf as/ *also written as*: $(-\infty;a-r)\cup(a+r;\infty)$

Vir die geval $|x-a|\geq r$ oftewel $|(x-a)-0|\geq r$ lyk die situasie soos volg:

For the case $|x-a|\geq r$ or just as well $|(x-a)-0|\geq r$ the situation is as follows:



$|x-a| \geq r$, dit is/ *that is* $|(x-a)-0| \geq r$ impliseer dat/ *implies that* $x-a \leq -r$ of/or $x-a \geq r$

waaruit volg/ *from which follows that* $x \leq a-r$ of/or $x \geq a+r$

ook geskryf as/ *also written as*: $(-\infty; a-r] \cup [a+r; \infty)$

Nog belangrike eienskappe van die absolute waarde-bewerking

Other important properties of the absolute value operation

$$|x| = \sqrt{x^2}$$

$$-|x| = -\sqrt{x^2}$$

$$|a||b| = |ab|$$

$$\frac{|a|}{|b|} = \left|\frac{a}{b}\right|$$

$$|-x+a| = |x-a|$$

As/ if $|x-a| = 0$ dan is/ then $x = a$

Voorbeeld/ example 1:

Bepaal/ *Determine* x sodat/ *such that* $2|3x-4| = 20$

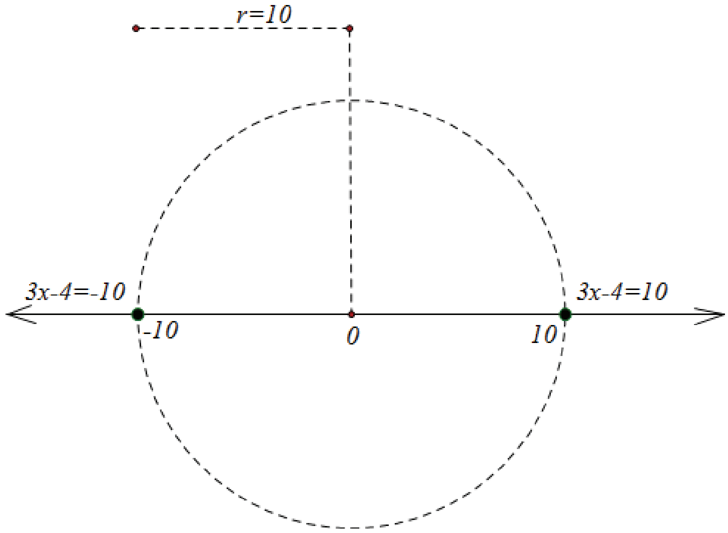
Oplossing (3 metodes)/ *Solution (3 methods):*

- Meetkundige definisie/ *Geometric definition*

$$2|3x-4|=20$$
$$\therefore |3x-4|=10$$
$$\therefore |(3x-4)-0|=10$$

Bepaal dus die waarde van $3x-4$ sodat $3x-4$ presies 10 eenhede vanaf die punt 0 op die getallelyn lê. Los dan op vir x./

So find the value of $3x-4$ *such that* $3x-4$ *lies precisely* 10 *units from the point* 0 . *Then solve for x.*



$\therefore 3x-4=-10$	of / *or*	$3x-4=10$
$\therefore x=-2$	of / *or*	$x=\frac{14}{3}$

- Formele definisie/ *Formal definition*:

$$2|3x-4|=20$$
$$\therefore |3x-4|=10$$

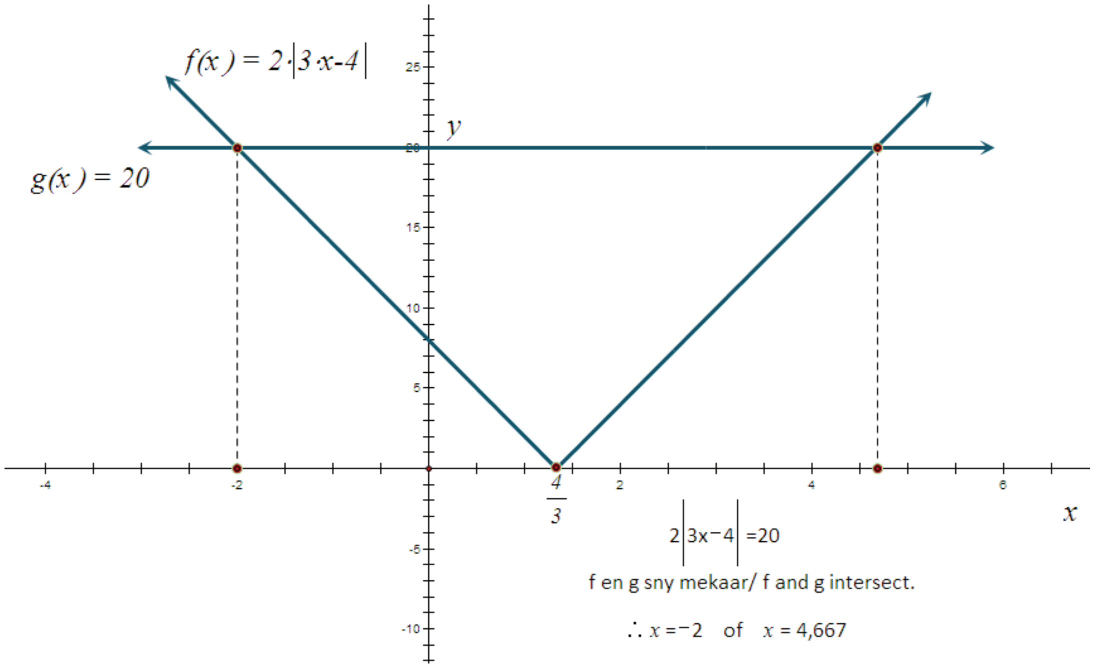
$$\therefore \begin{cases} 3x-4=10 & \text{as} \quad 3x-4\geq 0 \\ -(3x-4)=20 & \text{as} \quad 3x-4<0 \end{cases}$$

$$\therefore \begin{cases} 3x=14 & \text{as} \quad 3x\geq 4 \\ -3x+4=10 & \text{as} \quad 3x<4 \end{cases}$$

$$\therefore \begin{cases} x=\frac{14}{3} & \text{as} \quad x\geq \frac{4}{3} \\ -3x=6 & \text{as} \quad x<\frac{4}{3} \end{cases}$$

$$\therefore \begin{cases} x=4,667 & \text{as} \quad x\geq \frac{4}{3} \\ x=-2 & \text{as} \quad x<\frac{4}{3} \end{cases}$$

- Grafiese interpretasie/ *Graphical interpretation*



Voorbeeld/ *Example 2*:

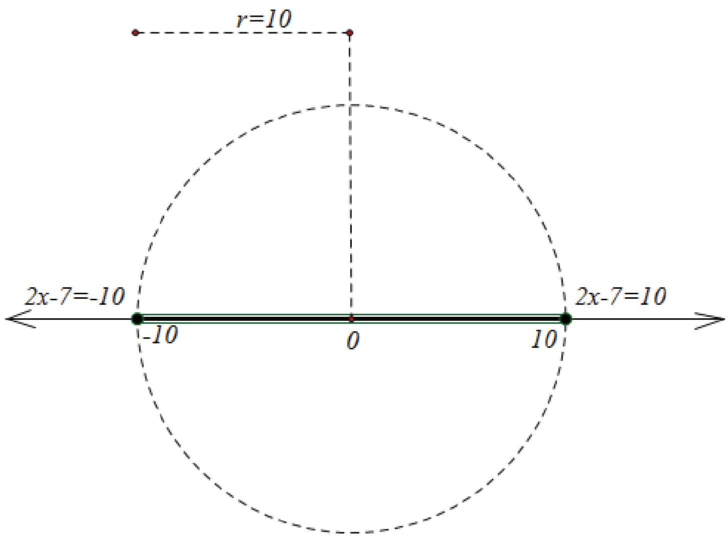
Los op vir/ Solve for x sodat/ such that $3|2x-7|\leq 30$

Oplossing (3 metodes)/ *Solution (3 methods):*

- Meetkundige definisie/ *Geometric definition*

$$3|2x-7|\leq 30$$
$$\therefore |2x-7|\leq 10$$
$$\therefore |(2x-7)-0|\leq 10$$

Bepaal dus die waarde van x sodat $2x-7$ binne 10 eenhede van die punt 0 op die getallelyn lê/
Find the value of x such that $2x-7$ lies less than 10 units from the point 0 on the number line



$$\therefore -10\leq 2x-7\leq 10$$
$$\therefore -3\leq 2x\leq 17$$
$$\therefore -\frac{3}{2}\leq x\leq \frac{17}{2}$$
$$\therefore x\in[-1,5;8,5]$$

- Formele definisie/ *Formal definition*:

$$3|2x-7|\leq 30$$

$$\therefore \begin{cases} 3(2x-7)\leq 30 & \text{as} \quad 2x-7\geq 0 \\ -3(2x-7)\leq 30 & \text{as} \quad 2x-7<0 \end{cases}$$

$$\therefore \begin{cases} 6x-21\leq 30 & \text{as} \quad 2x\geq 7 \\ -6x+21\leq 30 & \text{as} \quad 2x<7 \end{cases}$$

$$\therefore \begin{cases} 6x\leq 51 & \text{as} \quad x\geq \frac{7}{2} \\ -6x\leq 9 & \text{as} \quad x<\frac{7}{2} \end{cases}$$

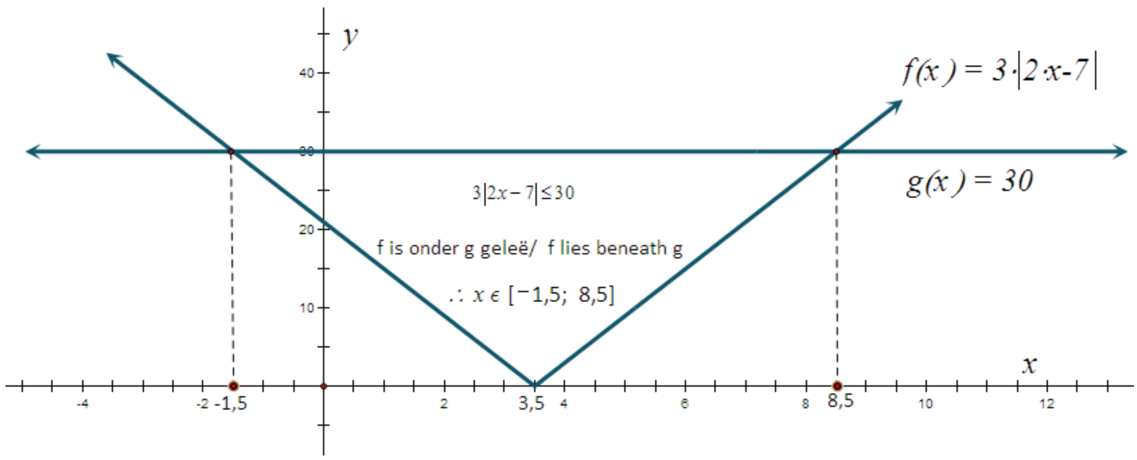
$$\therefore \begin{cases} x\leq \frac{51}{6} & \text{as} \quad x\geq \frac{7}{2} \\ x\geq -\frac{9}{6} & \text{as} \quad x<\frac{7}{2} \end{cases}$$

$$\therefore \begin{cases} x\leq 8{,}5 & \text{as} \quad x\geq 3{,}5 \\ x\geq -1{,}5 & \text{as} \quad x<3{,}5 \end{cases}$$

$$\therefore -1{,}5\leq x\leq 8{,}5$$

$$\therefore x\in[-1{,}5;8{,}5]$$

- Grafiese interpretasie/ *Graphical interpretation*



Voorbeeld/ *Example 3:*

Los op vir/ Solve for x sodat/ such that $-2|3x+4|<-6$

Oplossing (3 metodes)/ *Solution (3 methods):*

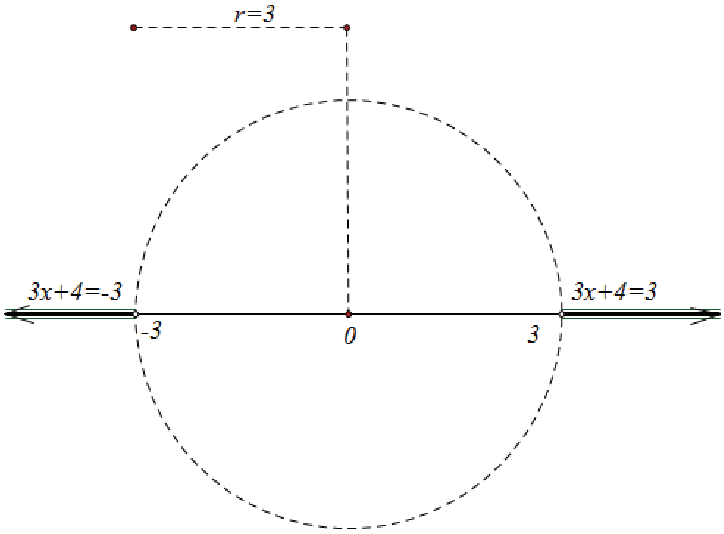
- Meetkundige definisie/ *Geometric definition*

$$-2|3x+4|<-6$$

$$\therefore |3x+4|>3$$

$$\therefore |(3x+4)-0|>3$$

Bepaal dus die waarde van x sodat $3x+4$ meer as 3 eenhede van die punt 0 op die getallelyn lê./ *Determine the value of x such that $3x+4$ lies more than 3 units from the point 0 on the number line.*



$\therefore 3x+4<-3$ of / *or* $3x+4>3$

$\therefore x<-\dfrac{7}{3}$ of / *or* $x>-\dfrac{1}{3}$

$\therefore \left(-\infty;-\dfrac{7}{3}\right)\cup\left(-\dfrac{1}{3};\infty\right)$

- Formele definisie/ *Formal definition*

$$-2|3x+4|<-6$$

$$\therefore |3x+4|>3$$

$$\therefore \begin{cases} 3x+4>3 & \text{as} \quad 3x+4\geq 0 \\ -(3x+4)>3 & \text{as} \quad 3x+4<0 \end{cases}$$

$$\therefore \begin{cases} 3x>-1 & \text{as} \quad 3x\geq -4 \\ -3x-4>3 & \text{as} \quad 3x<-4 \end{cases}$$

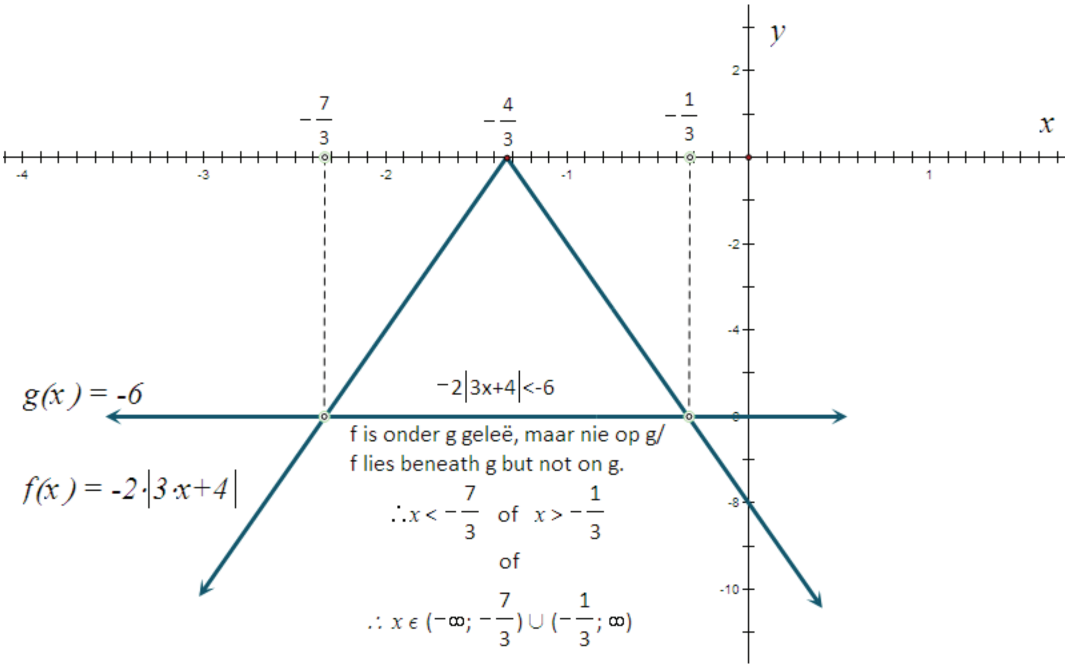
$$\therefore \begin{cases} 3x>-1 & \text{as} \quad x\geq -\dfrac{4}{3} \\ -3x>7 & \text{as} \quad x<-\dfrac{4}{3} \end{cases}$$

$$\therefore \begin{cases} x>-\dfrac{1}{3} & \text{as} \quad x\geq -\dfrac{4}{3} \\ x<-\dfrac{7}{3} & \text{as} \quad x<-\dfrac{4}{3} \end{cases}$$

$$\therefore x<-\frac{7}{3} \quad \text{of} \quad x>-\frac{1}{3}$$

$$\therefore x\in\left(-\infty;-\frac{7}{3}\right)\cup\left(-\frac{1}{3};\infty\right)$$

- Grafiese interpretasie/ *Graphical interpretation*



y
x
$-\frac{7}{3}$
$-\frac{4}{3}$
$-\frac{1}{3}$
$g(x) = -6$
$f(x) = -2\cdot|3x+4|$
$-2|3x+4|<-6$
f is onder g geleë, maar nie op g/
f lies beneath g but not on g.
$\therefore x < -\frac{7}{3}$ of $x > -\frac{1}{3}$
of
$\therefore x \in (-\infty; -\frac{7}{3}) \cup (-\frac{1}{3}; \infty)$

Oefening 5.2

1. Los die onbekende op:

Exercise 5.2

1. Solve for the unknown:

1.1 $3|2-k|=0$

1.2 $-2|2t+1|=6$

1.3 $\left|\frac{1}{2}r+\frac{1}{2}\right|=|-2|$

2. Los op vir die onbekende en stel die oplossing op 'n getallelyn voor.

2. Solve for the unknown and represent the solutions on a number line.

2.1 $|x+2| \leq 2$

2.2 $-3|2x-5| < -9$

2.3 $-6\left|\dfrac{2-3x}{4}\right| < -6$

3. Skryf die volgende in absolute waarde-notasie:

3. Skryf die volgende in absolute waarde-notasie:

3.1 x is minder as 3 eenhede vanaf 7

3.1 x is minder as 3 eenhede vanaf 7

3.2 t is nie meer as 5 eenhede vanaf 8

3.2 t is nie meer as 5 eenhede vanaf 8

3.3 y lê tussen -3 en 3

3.3 y lê tussen -3 en 3

3.4 Die afstand tussen 6 en m is 4

3.4 Die afstand tussen 6 en m is 4

5.3 Limiete en kontinuïteit/ *Limits and continuity*

Dikwels stel ons daarin belang om te weet wat die waarde is wat 'n funksie aanneem wanneer die onafhanklike veranderlike baie groot negatief raak, of baie groot positief raak, of wanneer die onafhanklike 'n sekere waarde aanneem.

Often we are interested in the value which is assumed by a function when the independent variable becomes a very large negative value, or a very large positive value, or when the independent variable assumes a particular value.

Dit is nie in alle gevalle moontlik om gewoon die onafhanklike veranderlike in die funksie te vervang en dan die funksiewaarde te bereken nie.

It is not always possible to simply substitute the independent variable into the function equation and then obtain the the function value.

Vervolgens kyk ons na funksies wie se gedrag slegs met behulp van limiete volledig beskryf kan word.

Next, we consider functions of which the behaviour can only be fully described in terms of limits.

Voorbeeld 1

Example 1

Beskou die funksie $f(x)=\dfrac{2x+1}{x-1}-\dfrac{2}{x-3}$

Consider the function
$f(x)=\dfrac{2x+1}{x-1}-\dfrac{2}{x-3}$

1. Bepaal $f(2)$, dit is die funksiewaarde in die punt waar $x=2$.

1. *Determine* $f(2)$ *, that is the function value in the point where* $x=2$.

2. Die funksiewaarde in die punt waar $x=2$ gee egter geen inligting oor die gedrag van die funksie baie naby aan die punt waar $x=2$ nie.

   Laat ons die gedrag van die funksie ondersoek in die omgewing van hierdie punt.

2. *The function value in the point where* $x=2$ *provides no information about the behaviour of the function close to the point where* $x=2$.

   Let us now investigate the behaviour of the function in the vicinity or proximity of this point.

Voltooi die volgende tabel: *Complete the following table:*

As/if $x<2$		As/if $x>2$	
x	$f(x)=\frac{2x+1}{x-1}-\frac{2}{x-3}$	x	$f(x)=\frac{2x+1}{x-1}-\frac{2}{x-3}$
1		2,001	
1,5		2,01	
1,9		2,1	
1,99		2,5	
1,999		3	

2.1.1 Voltooi: Wanneer x vanaf die linkerkant streef na 2, dan streef die funksiewaarde na

....

2.1.1 Complete: When x tends from the left side to 2, then the function value tends to

....

2.1.2 Skryf dit in limiet-notasie:

$$\lim_{x\to 2^-}\left(\frac{2x+1}{x-1}-\frac{2}{x-3}\right)=\ldots\ldots\ldots\ldots$$

of korter:

$$\lim_{x\to 2^-} f(x)=\ldots\ldots\ldots\ldots$$

2.1.2 Write this in limit notation:

$$\lim_{x\to 2^-}\left(\frac{2x+1}{x-1}-\frac{2}{x-3}\right)=\ldots\ldots\ldots\ldots$$

or shorter:

$$\lim_{x\to 2^-} f(x)=\ldots\ldots\ldots\ldots$$

2.2.1 Voltooi: Wanneer x vanaf die regterkant streef na 2, dan streef die funksiewaarde na

....

2.2.1 Complete: When x tends from the right side to 2, then the function value tends to

....

2.2.2 Skryf dit in limiet-notasie:

$$\lim_{x\to 2^+} f(x)=\ldots\ldots\ldots\ldots$$

2.2.2 Write this in limit notation:

$$\lim_{x\to 2^+} f(x)=\ldots\ldots\ldots\ldots$$

2.3 Vergelyk 2.1.2 en 2.2.2.

2.3 Compare 2.1.2 and 2.2.2.

Voltooi:

$\lim_{x \to 2^-} f(x) \ldots\ldots\ldots \lim_{x \to 2^+} f(x)$

Complete:

$\lim_{x \to 2^-} f(x) \ldots\ldots\ldots \lim_{x \to 2^+} f(x)$

Wanneer die linkerlimiet en die regterlimiet van 'n funksie dieselfde waarde aanneem in 'n punt, dan sê ons die funksie het 'n limiet in daardie punt.

When the left-sided limit and the right-sided limit of a function assumes the same value in a point, then we say that the function has a limit in that point.

Beskou 2.1.2 en 2.2.2 en voltooi:

Consider 2.1.2 and 2.2.2 and complete:

Omdat $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = \ldots\ldots\ldots\ldots$ kan ons skryf:

$\lim_{x \to 2} f(x) = \ldots\ldots\ldots\ldots$

Because $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = \ldots\ldots\ldots\ldots$ *we may write:*

$\lim_{x \to 2} f(x) = \ldots\ldots\ldots\ldots$

Die limiet van 'n funksie in 'n bepaalde punt bestaan slegs as beide die linkerlimiet en die regterlimiet in daardie punt bestaan en biede die linkerlimiet en die regterlimiet dieselfde waarde het.

The limit of a function in a particular point only exists provided that both the left-sided limit and the right-sided limit exist in that point and that they are equal in value.

In hierdie geval het ons dat die funksiewaarde in die punt waar $x = 2$ en die limiet van die funksie wanneer x van albei kante streef na 2 dieselfde waarde lewer:

$\lim_{x \to 2} f(x) = f(2)$

In this case we have that both the function value in the point where $x = 2$ and the limit of the function when x tends from both sides to 2 yield the same value:

$\lim_{x \to 2} f(x) = f(2)$

Hierdie spesiale sameloop van omstandighede impliseer dat die funksie f **kontinu** (aaneenlopend, sonder spronge of onderbrekings) is in die punt waar $x = 2$.

*This special set of circumstances implies that f is **continuous** (solid curve without gaps or breaks or jumps) in the point where $x = 2$.*

3. Laat die gedrag van die funksie ondersoek waar x baie groot negatief en waar x baie groot positief word.

   Voltooi die volgende tabel:

3. *Let us now investigate the behaviour of the function where x becomes a very large negative value and where x becomes a very large positive value.*

   Complete the following table:

As/if $x \to -\infty$		As/if $x \to \infty$	
x	$f(x)=\frac{2x+1}{x-1}-\frac{2}{x-3}$	x	$f(x)=\frac{2x+1}{x-1}-\frac{2}{x-3}$
-10		10	
-100		100	
-1000		1000	
$-10\,000$		10 000	

3.1.1 Voltooi: Wanneer x streef na negatief oneindig, dan streef die funksiewaarde na

....

3.1.1 Complete: When x tends to negative infinity, then the function value tends to

....

3.1.2 Skryf dit in limiet-notasie:

$$\lim_{x\to-\infty}\left(\frac{2x+1}{x-1}-\frac{2}{x-3}\right)=\text{.....................}$$

of korter:

$$\lim_{x\to-\infty} f(x)=\text{.....................}$$

3.1.2 Rewrite this in limit notation:

$$\lim_{x\to-\infty}\left(\frac{2x+1}{x-1}-\frac{2}{x-3}\right)=\text{.....................}$$

or shorter:

$$\lim_{x\to-\infty} f(x)=\text{.....................}$$

3.2.1 Voltooi: Wanneer x streef na positief oneindig, dan streef die funksiewaarde na

....

3.2.1 Complete: When x tends to positive infinity, then the function value tends to

....

3.2.2 Skryf dit in limiet-notasie:

$$\lim_{x\to\infty} f(x)=\text{.....................}$$

3.2.2 Write this in limit notation:

$$\lim_{x\to\infty} f(x)=\text{.....................}$$

Uit die limiete $\lim_{x\to-\infty} f(x)$ en $\lim_{x\to\infty} f(x)$ kan ons dikwels die **horisontale asimptote** van 'n funksie vind.

*From the limits $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\infty} f(x)$ we can often find the **horizontal asymptotes** of a function.*

4. Bepaal $f(3)$, dit is die funksiewaarde in die punt waar $x=3$.

4. Determine $f(3)$, that is the function value where $x=3$.

5. Die funksiewaarde in die punt waar $x=3$ sou selfs al het dit bestaan, geen inligting gegee het oor die gedrag van die funksie in die omgewing van die punt waar $x=3$ is nie.

Laat ons die gedrag van die funksie ondersoek in die omgewing van hierdie punt, waar die funksie ongedefinieerd is.

Voltooi die volgende tabel:

5. The function value in the point where $x=3$ would not, even if it existed, provide any information about the behaviour of the function in the vicinity of the point where $x=3$.

Let us now investigate the behaviour of the function in the vicinity of this point, where the function is undefined.

Complete the following table:

As/if $x<3$		As/if $x>3$	
x	$f(x)=\frac{2x+1}{x-1}-\frac{2}{x-3}$	x	$f(x)=\frac{2x+1}{x-1}-\frac{2}{x-3}$
2		3,001	
2,5		3,01	
2,9		3,1	
2,99		3,5	
2,999		4	

5.1.1 Voltooi: Wanneer x vanaf die linkerkant streef na 3, dan streef die funksiewaarde na

....

5.1.1 Complete: When x tends to 3 from the left side, then the function value tends to

....

5.1.2 Skryf dit in limiet-notasie:

$$\lim_{x \to 3^-} \left(\frac{2x+1}{x-1} - \frac{2}{x-3} \right) = \ldots\ldots\ldots\ldots\ldots\ldots$$

of korter:

$$\lim_{x \to 3^-} f(x) = \ldots\ldots\ldots\ldots\ldots\ldots$$

5.2.1 Voltooi: Wanneer x vanaf die regterkant streef na 3, dan streef die funksiewaarde na

....

5.2.2 Skryf dit in limiet-notasie:

$$\lim_{x \to 3^+} f(x) = \ldots\ldots\ldots\ldots\ldots\ldots$$

5.3 Vergelyk 5.1.2 en 5.2.2.

Voltooi:

$$\lim_{x \to 3^-} f(x) \ldots\ldots\ldots \lim_{x \to 3^+} f(x)$$

Wanneer die linkerlimiet en die regterlimiet van 'n funksie nie dieselfde waarde aanneem in 'n punt nie, dan sê ons die funksie het nie 'n limiet in daardie punt nie.

Beskou 5.1.2 en 5.2.2 en voltooi:

Omdat $\lim_{x \to 3^-} f(x) \neq \lim_{x \to 3^+} f(x)$ kan ons skryf:

$\lim_{x \to 3} f(x)$ bestaan nie.

Opmerking: Aangesien die linkerlimiet en regterlimiet elk ook nie na 'n vaste reële waarde streef nie, bestaan die linker- en regterlimiet in hierdie geval ook nie.

5.1.2 Write this in limit notation:

$$\lim_{x \to 3^-} \left(\frac{2x+1}{x-1} - \frac{2}{x-3} \right) = \ldots\ldots\ldots\ldots\ldots\ldots$$

or shorter:

$$\lim_{x \to 3^-} f(x) = \ldots\ldots\ldots\ldots\ldots\ldots$$

5.2.1 Complete: When x tends to 3 from the right side, then the function value tends to

....

5.2.2 Write this in limit notation:

$$\lim_{x \to 3^+} f(x) = \ldots\ldots\ldots\ldots\ldots\ldots$$

5.3 Compare 5.1.2 and 5.2.2.

Complete:

$$\lim_{x \to 3^-} f(x) \ldots\ldots\ldots \lim_{x \to 3^+} f(x)$$

When the left-sided limit and the right-sided limit of a function does not assume the same value in a point, then we say that the function does not have a limit in that point.

Consider 5.1.2 and 5.2.2 and complete:

Because $\lim_{x \to 3^-} f(x) \neq \lim_{x \to 3^+} f(x)$ *we may write:*

$\lim_{x \to 3} f(x)$ *does not exist.*

***Remark**: Because the left-sided limit and the right-sided limit each does not tend to a fixed real value, neither the left-sided limit nor the right-sided limit exists in this case .*

Ons noem die vertikale lyn met vergelyking $x = 3$ 'n vertikale asimptoot van die funksie.

We refer to the vertical line with equation $x = 3$ ***as a vertical asymptote of the function.***

Die limiet van 'n funksie in 'n bepaalde punt bestaan slegs as beide die linkerlimiet en die regterlimiet in daardie punt bestaan en dieselfde waarde besit.

The limit of a function in a particular point only exist provided that the left-sided limit and the right-sided limit both exist in that point and that they have the same value.

In hierdie geval het ons dat die funksiewaarde in die punt waar $x = 3$ nie bestaan nie en dat die limiet van die funksie wanneer x van albei kante streef na 3 ook nie bestaan nie.

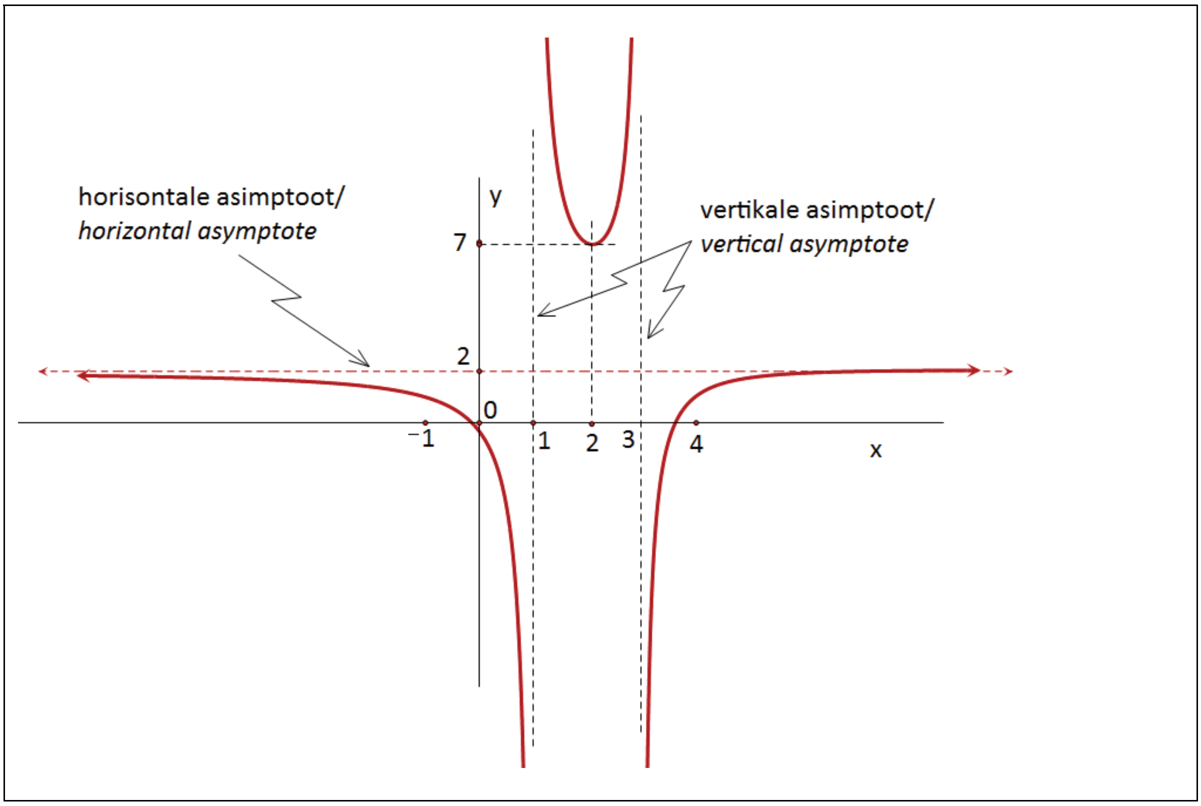
In this case we have that the function value in the point where $x = 3$ does not exist and that the limit of the function when x tends to 3 from both sides does not exist.

Hierdie spesiale sameloop van omstandighede impliseer dat die funksie f **diskontinu** (nie-aaneenlopend, met 'n sprong of onderbreking) is in die punt waar $x = 3$.

This special set of circumstances implies that the function f is ***discontinuous*** *(not solid, with a gap or a jump) in the point where $x = 3$.*

Vir duidelikheid toon ek hieronder 'n rekenaarvoorstelling van die funksie wat ons ondersoek het:

For clarity I show a computer-generated representation of the function we have investigated:



Voorbeeld 2

Beskou die funksie

$$f(x)=\begin{cases} x^2 & \text{as} \quad x<1 \\ 2 & \text{as} \quad x=1 \\ x & \text{as} \quad x>1 \end{cases}$$

3. Bepaal $f(1)$, dit is die funksiewaarde in die punt waar $x=1$.

4. Die funksiewaarde in die punt waar $x=1$ gee egter geen inligting oor die gedrag van die funksie baie naby aan die punt waar $x=1$ nie.

   Laat ons die gedrag van die funksie ondersoek in die omgewing van hierdie punt.

   Voltooi die volgende tabel:

Example 2

Consider the function

$$f(x)=\begin{cases} x^2 & \text{if} \quad x<1 \\ 2 & \text{if} \quad x=1 \\ x & \text{if} \quad x>1 \end{cases}$$

2. *Determine* $f(1)$ *, that is the function value in the point where* $x=1$.

4. *The function value in the point where* $x=1$ *provides no information about the behaviour of the function close to the point where* $x=1$.

   Let us now investigate the behaviour of the function in the vicinity or proximity of this point.

   Complete the following table:

As/*if* $x<1$		As/*if* $x>1$	
x	$f(x)=\begin{cases} x^2 & \text{as/if} \quad x<1 \\ 2 & \text{as/if} \quad x=1 \\ x & \text{as/if} \quad x>1 \end{cases}$	x	$f(x)=\begin{cases} x^2 & \text{as/if} \quad x<1 \\ 2 & \text{as/if} \quad x=1 \\ x & \text{as/if} \quad x>1 \end{cases}$
0		1,001	
0,5		1,01	
0,9		1,1	
0,99		1,5	
0,999		2	

2.1.1 Voltooi: Wanneer x vanaf die linkerkant streef na 1, dan streef die funksiewaarde na

....

2.1.1 Complete: When x tends from the left side to 1, then the function value tends to

....

2.1.2 Skryf dit in limiet-notasie:

$\lim\limits_{x\to1^-} f(x) =$ ....

2.1.2 Write this in limit notation:

$\lim\limits_{x\to1^-} f(x) =$ ....

2.2.1 Voltooi: Wanneer x vanaf die regterkant streef na 1, dan streef die funksiewaarde na

....

2.2.1 Complete: When x tends from the right side to 1, then the function value tends to

....

2.2.2 Skryf dit in limiet-notasie:

$\lim\limits_{x\to1^+} f(x) =$ ....

2.2.2 Write this in limit notation:

$\lim\limits_{x\to1^+} f(x) =$ ....

2.3 Vergelyk 2.1.2 en 2.2.2.

Voltooi:

$\lim\limits_{x\to1^-} f(x)$ .... $\lim\limits_{x\to1^+} f(x)$

2.3 Compare 2.1.2 and 2.2.2.

Complete:

$\lim\limits_{x\to1^-} f(x)$ .... $\lim\limits_{x\to1^+} f(x)$

Wanneer die linkerlimiet en die regterlimiet van 'n funksie dieselfde waarde aanneem in 'n punt, dan sê ons die funksie het 'n limiet in daardie punt.

When the left-sided limit and the right-sided limit of a function assumes the same value in a point, then we say that the function has a limit in that point.

Beskou 2.1.2 en 2.2.2 en voltooi:

Consider 2.1.2 and 2.2.2 and complete:

Omdat $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = ...$ kan ons skryf:

$\lim_{x\to 1} f(x) = ...$

Because $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = ...$ *we may write:*

$\lim_{x\to 1} f(x) = ...$

Die limiet van 'n funksie in 'n bepaalde punt bestaan slegs as beide die linkerlimiet en die regterlimiet in daardie punt bestaan en biede die linkerlimiet en die regterlimiet dieselfde waarde het.

The limit of a function in a particular point only exists provided that both the left-sided limit and the right-sided limit exist in that point and that they are equal in value.

In hierdie geval het ons dat die funksiewaarde in die punt waar $x = 1$ en die limiet van die funksie wanneer x van albei kante streef na 1 verskillende waardes lewer:

$\lim_{x\to 1} f(x) \neq f(1)$

In this case we have that both the function value in the point where $x = 1$ *and the limit of the function when* x *tends from both sides to 2 yield different values:*

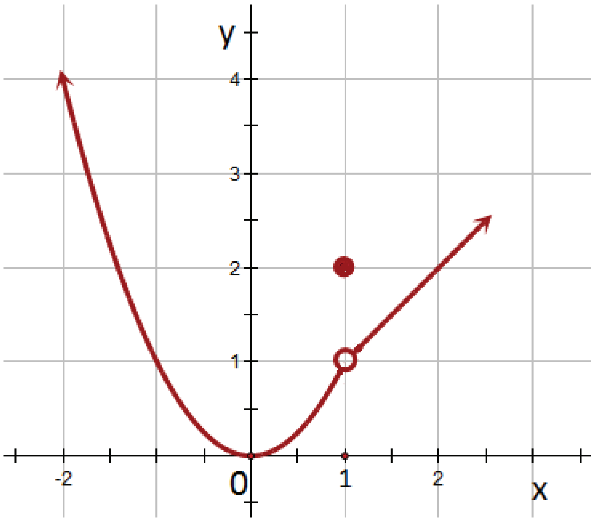
$\lim_{x\to 1} f(x) \neq f(1)$

Hierdie spesiale sameloop van omstandighede impliseer dat die funksie f **diskontinu** is in die punt waar $x = 1$.

This special set of circumstances implies that f *is* ***discontinuous*** *in the point where* $x = 1$.

Vir duidelikheid toon ek hieronder 'n rekenaarvoorstelling van die funksie wat ons ondersoek het:

For clarity I show a computer-generated representation of the function we have investigated:



Voorbeeld 3

Beskou die funksie $f(x)=\dfrac{8-x^3}{2-x}$

1. Bepaal $f(2)$, dit is die funksiewaarde in die punt waar $x=2$.

2. Laat ons die gedrag van die funksie ondersoek in die omgewing van hierdie punt.

   Voltooi die volgende tabel:

Example 3

Consider the function $f(x)=\dfrac{8-x^3}{2-x}$

1. *Determine* $f(2)$ *, that is the function value in the point where* $x=2$.

2. *Let us now investigate the behaviour of the function in the vicinity or proximity of this point.*

   Complete the following table:

As/*if* $x<2$		As/*if* $x>2$	
x	$f(x)=\dfrac{8-x^3}{2-x}$	x	$f(x)=\dfrac{8-x^3}{2-x}$
1		2,001	
1,5		2,01	
1,9		2,1	
1,99		2,5	
1,999		3	

2.1.1 Voltooi: Wanneer x vanaf die linkerkant streef na 2, dan streef die funksiewaarde na

....

2.1.1 Complete: When x tends from the left side to 2, then the function value tends to

....

2.1.2 Skryf dit in limiet-notasie:

$\lim\limits_{x\to 2^-} f(x) =$ ....

2.1.2 Write this in limit notation:

$\lim\limits_{x\to 2^-} f(x) =$ ....

2.2.1 Voltooi: Wanneer x vanaf die regterkant streef na 2, dan streef die funksiewaarde na

....

2.2.1 Complete: When x tends from the right side to 2, then the function value tends to

....

2.2.2 Skryf dit in limiet-notasie:

$\lim\limits_{x\to 2^+} f(x) =$ ....

2.2.2 Write this in limit notation:

$\lim\limits_{x\to 2^+} f(x) =$ ....

2.3 Vergelyk 2.1.2 en 2.2.2.

Voltooi:

$\lim\limits_{x\to 2^-} f(x)$ .... $\lim\limits_{x\to 2^+} f(x)$

2.3 Compare 2.1.2 and 2.2.2.

Complete:

$\lim\limits_{x\to 2^-} f(x)$ .... $\lim\limits_{x\to 2^+} f(x)$

Wanneer die linkerlimiet en die regterlimiet van 'n funksie dieselfde waarde aanneem in 'n punt, dan sê ons die funksie het 'n limiet in daardie punt.

When the left-sided limit and the right-sided limit of a function assumes the same value in a point, then we say that the function has a limit in that point.

Beskou 2.1.2 en 2.2.2 en voltooi:

Consider 2.1.2 and 2.2.2 and complete:

Omdat $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^+} f(x) = ...$ kan ons skryf:

$\lim\limits_{x\to 2} f(x) = ...$

Because $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^+} f(x) = ...$ *we may write:*

$\lim\limits_{x\to 2} f(x) = ...$

Die limiet van 'n funksie in 'n bepaalde punt bestaan slegs as beide die linkerlimiet en die regterlimiet in daardie punt bestaan en beide die linkerlimiet en die regterlimiet dieselfde waarde het.

LET DAAROP dat die funksie nie in daardie punt gedefinieer hoef te wees nie; die limiet bestaan in elk geval as beide die linkerlimiet en die regterlimiet dieselfde waarde het.

The limit of a function in a particular point only exists provided that both the left-sided limit and the right-sided limit exist in that point and that they are equal in value.

NOTE that the function need not be defined in that point; the limit exists whenever both the left-sided limit and the right-sided are equal in value.

In hierdie geval het ons dat die funksiewaarde in die punt waar $x = 2$ en die limiet van die funksie wanneer x van albei kante streef na 2 nie dieselfde waarde lewer nie; **die funksiewaarde bestaan dan nie eers nie**:

$\lim\limits_{x\to 2} f(x) \neq f(2)$

In this case we have that both the function value in the point where $x = 2$ and the limit of the function when x tends from both sides to 2 does not yield the same value; ***in fact, the function value does not exist****:*

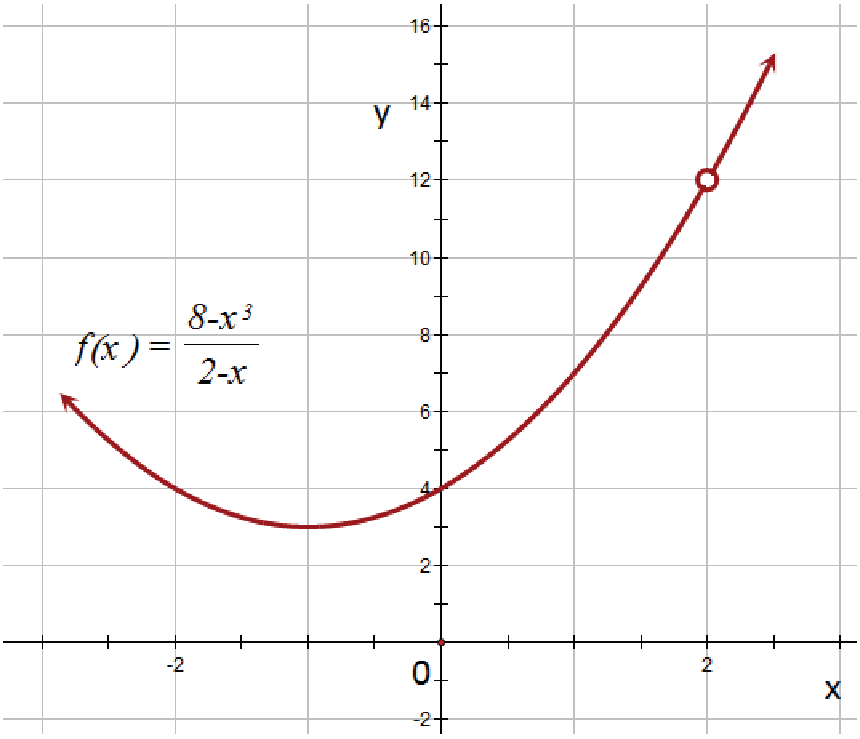
$\lim\limits_{x\to 2} f(x) \neq f(2)$

Hierdie spesiale sameloop van omstandighede impliseer dat die funksie f **diskontinu** is in die punt waar $x = 2$.

This special set of circumstances implies that f is ***discontinuous*** *in the point where $x = 2$.*

Vir duidelikheid toon ek hieronder 'n rekenaarvoorstelling van die funksie wat ons ondersoek het:

For clarity I show a computer-generated representation of the function we have investigated:



Let egter daarop dat die limiet van hierdie funksie ook soos volg verkry kan word:

Note, however, that the limit of this function may also be obtained in the following way:

$$
\begin{aligned}
&\lim_{x\to 2}\frac{8-x^3}{2-x}\\
&=\lim_{x\to 2}\frac{(2-x)(4+2x+x^2)}{2-x}\\
&=\lim_{x\to 2}\frac{\cancel{(2-x)}(4+2x+x^2)}{\cancel{(2-x)}}\\
&=\lim_{x\to 2}(4+2x+x^2)\\
&=4+2(2)+2^2\\
&=12
\end{aligned}
$$

Hierdie tipe limiet, **waar die noemer wat nul sou word tydens direkte vervanging verwyder kan word deur faktorisering en deling**, word 'n **verwyderbare diskontinuït** genoem.

This type of limit, where the numerator which would become zero during direct substitution may be removed by factorization and division, is called a removable discontinuity.

Hulle word gekenmerk deur dat die vorm $\frac{0}{0}$ ontstaan wanneer die waarde waarheen die onafhanklike veranderlike streef, direk in vervang word.

Die uitdeelbewerking is toelaatbaar, aangesien die noemer streef na nul, maar nie nul word nie.

They are characterized by the form $\frac{0}{0}$ which appears when the value which the independent variable is approaching, is substituted directly into the function.

The division operation here is permissible, because the denominator only tends to zero – it does not actually "become" zero.

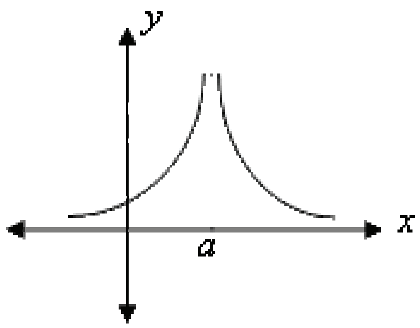
Oefening 5.3

1. Ondersoek al die voorwaardes vir kontinuïteit en spesifiseer watter van die voorwaardes verbreek word by $x = a$ vir elke funksie.

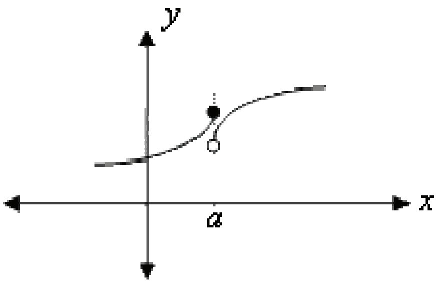
Exercise 5.3

1. Investigate all the conditions for continuity and specify which of the conditions do not hold at $x = a$ for each function*:*

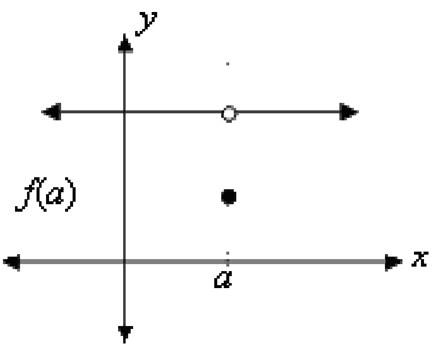
1.1

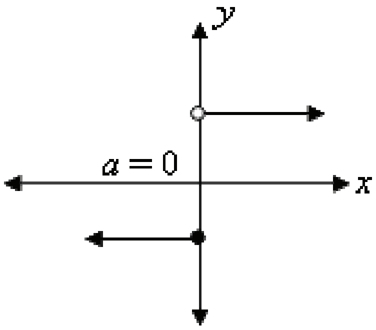


1.2



1.3





2. Gebruik die voorwaardes van kontinuïteit om aan te toon dat die volgende funksie kontinu is by die gegewe punt: $g(t)=\dfrac{t-3}{9t}$, $t=-3$

2. Use the conditions for continuity and show that the following function is continuous in the specified point: $g(t)=\dfrac{t-3}{9t}$,

5.4 Berekening van sekere limiete/ *Calculation of certain limits*

Direkte metode berekening van limiete kom daarop neer dat ons 'n tabelmetode vermy en eerder direkte vervanging probeer.

Wanneer direkte vervanging nie werk nie, toets ons of ons een van die volgende twee spesiale soort limiete het.

Direct methods for calculating limits boil down to avoiding a table method but attempting instead direct substitution.

When direct substitution fails we test whether or not we are dealing with one of the following special cases .

5.4.1 Die spesiale geval $\frac{0}{0}$

5.4.1 The special case $\frac{0}{0}$

Ons het hierdie soort aan die einde van Leergedeelte 5.3 teëgekom.

We encountered this type at the end of Study Section 5.3.

Voorbeeld/ *Example*

Bereken/ Calculate $\lim\limits_{p\to-5} \frac{p^2-25}{p+5}$

Oplossing/ *Solution*

$\lim\limits_{p\to-5} \frac{p^2-25}{p+5}$ vervanging lewer / *substitution yield* $\frac{(-5)^2-25}{-5+5}=\frac{0}{0}$

$$
\begin{aligned}
&= \lim_{p\to-5} \frac{(p-5)\cancel{(p+5)}}{\cancel{(p+5)}} \\
&= \lim_{p\to-5} (p-5) \\
&= -5-5 \\
&= -10
\end{aligned}
$$

5.4.2 Die spesiale geval $\frac{\infty}{\infty}$

5.4.2 The special case $\frac{\infty}{\infty}$

By hierdie soort deel ons in elke term deur die hoogste mag van die veranderlike wat voorkom.

When dealing with this type, we divide each term by the highest power of the variable which occurs.

Voorbeeld/ *Example*

Bereken/ Calculate $\lim\limits_{t\to\infty} \frac{t^2}{t^2-1}$

Oplossing/ *Solution*

$\lim_{t\to\infty} \frac{t^2}{t^2-1}$ vervanging lewer / *substitution yield* $\frac{\infty}{\infty}$

$$= \lim_{t\to\infty} \frac{\left(\frac{t^2}{t^2}\right)}{\left(\frac{t^2}{t^2}-\frac{1}{t^2}\right)}$$

$= \lim_{t\to\infty} \frac{1}{\left(1-\frac{1}{t^2}\right)}$ maar/ *but* $\frac{1}{t^n} \to 0$ as / *if* $t \to \infty$ en / *and* $n > 0$

$$= \frac{1}{(1-0)}$$

$$= 1$$

Oefening 5.4 ***Exercise 5.4***

Bereken die volgende direk (geen tabel) *Calculate the following directly (no table)*

1. $\lim_{x\to 0} \frac{10x^2}{x}$

2. $\lim_{x\to -1} \frac{x^2-1}{x+1}$

3. $\lim\limits_{t \to 3} \dfrac{t^2 - 5t + 6}{t - 3}$

4. $\lim\limits_{k \to -4} \dfrac{64 + k^3}{4 + k}$

5. $\lim\limits_{x \to \infty} \dfrac{10x^2}{x}$

6. $\lim\limits_{x \to \infty} \dfrac{3x^2 - 5x + 7}{2x^2 - x}$

7. $\lim_{t\to\infty} \frac{2t+5}{3t+2}$

8. $\lim_{m\to\infty} \frac{m^3-27}{m-3}$

5.5 Die epsilon-delta definisie van 'n limiet/ *The epsilon-delta definition of a limit*

Die behoefte bestaan om 'n limiet op so 'n wyse formeel te definieer, dat ons wiskundig kan bewys dat 'n bepaalde funksie f 'n limiet L besit wanneer $x \to a$.

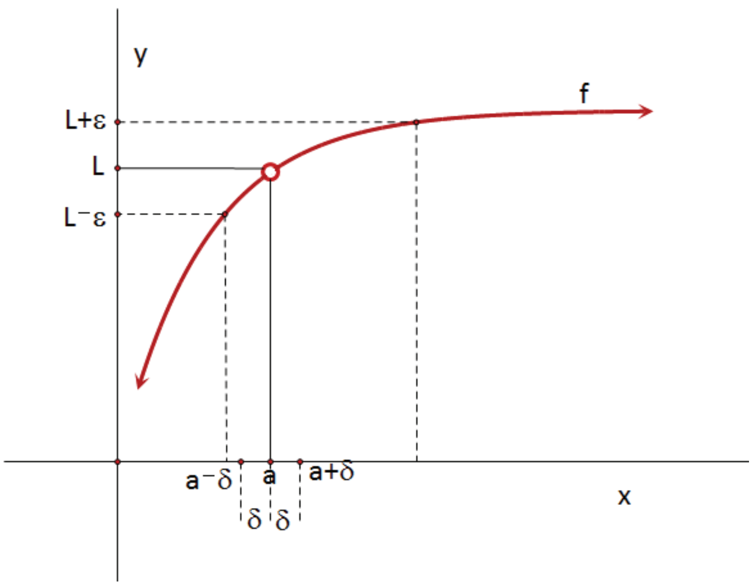
In die definisie dui ε op 'n afstand vanaf die punt L op die y-as.

In die definisie dui δ op 'n afstand vanaf die punt a op die x-as.

The need arise to define a limit formally in such a way, that we can mathematically prove that a given function f has a limit L when $x \to a$.

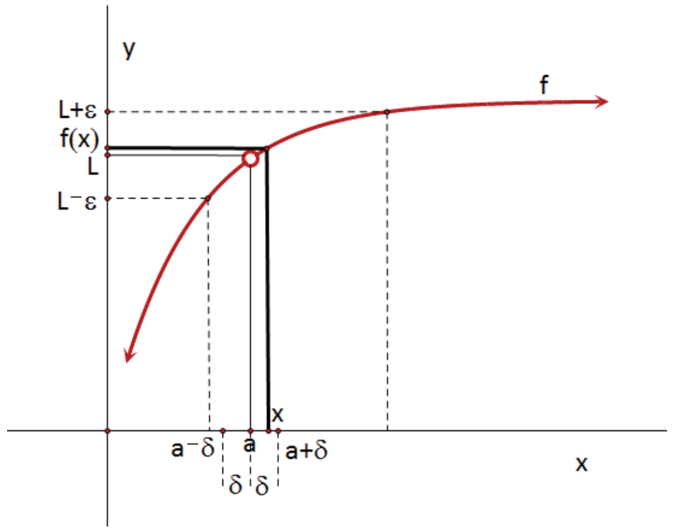
In the definition ε refers to a distance from L on the y-axis.

In the definition δ refers to a distance from a on the x-axis.



Die definisie stel kortliks dat as ons 'n sekere interval $(L-\varepsilon; L+\varepsilon)$ op die y-as neem en dit in die funksie f op die x-as afbeeld en ons om die punt a op die x-as 'n interval $(a-\delta; a+\delta)$ kan kry sodat elke x in hierdie interval se funksiewaarde nader aan L lê as ε, dan geld dat $f(x) \to L$ as $x \to a$.

The definition states in brief that if we take a certain interval $(L-\varepsilon; L+\varepsilon)$ on the y-axis and we project it in the function f to the x-asis and we can get around the point a on the x-axis an interval $(a-\delta; a+\delta)$ such that the function value of every x inside this interval lies closer to L than ε, then it holds that $f(x) \to L$ when $x \to a$.



Die bewering $\lim\limits_{x \to a} f(x) = L$ beteken dus dat L die limiet van die funksie f is indien $f(x)$ willekeurig naby aan L gebring kan word deur x naby genoeg aan a te kies.

The statement $\lim\limits_{x \to a} f(x) = L$ means that L is the limit of the function f if $f(x)$ can be made arbitrarily close to L by choosing x close enough to a.

Definisie van 'n limiet

Laat $f(x)$ gedefinieer wees vir alle x in 'n oop interval wat die getal a bevat, met die moontlike uitsluiting dat $f(x)$ nie gedefineer is in die punt a nie.

Ons skryf $\lim\limits_{x \to a} f(x) = L$

as ons vir enige positiewe getal ε 'n bybehorende positiewe getal δ kan vind sodat $|f(x) - L| < \varepsilon$ wanneer x voldoen aan $0 < |x - a| < \delta$.

Definition of a limit

Let $f(x)$ be defined for all x in some open interval containing the number a, with the possible exception that $f(x)$ may not be defined at a.

We shall write $\lim\limits_{x \to a} f(x) = L$

if given any positive number ε we can find a positive number δ such that $|f(x) - L| < \varepsilon$ if x satisfies $0 < |x - a| < \delta$.

Voorbeeld/ *Example*

Bewys dat/ Prove that $\lim\limits_{x\to 2}(3x-5)=1$

Oplossing/ *Solution*

Ons moet toon dat ons vir enige positiewe getal ε 'n positiewe getal δ kan vind sodat

We must show that given any positive number ε we can find a positive number δ such that

$$\left|\underbrace{(3x-5)}_{f(x)}-\underbrace{1}_{L}\right|<\varepsilon \qquad \text{as / } if \quad 0<\left|x-\underbrace{2}_{a}\right|<\delta$$

Maar ons kan dit herskryf as / *But we may rewrite this as*

$$|3x-6|<\varepsilon \qquad \text{as / } if \quad 0<|x-2|<\delta$$

$$\therefore 3|x-2|<\varepsilon \qquad \text{as / } if \quad 0<|x-2|<\delta$$

$$\therefore |x-2|<\frac{\varepsilon}{3} \qquad \text{as / } if \quad 0<|x-2|<\delta$$

As ons $\delta=\frac{\varepsilon}{3}$ kies word die regterkant van die bewering: / *If we choose* $\delta=\frac{\varepsilon}{3}$ *then the right – hand side the statement becomes*:

$0<|x-2|<\frac{\varepsilon}{3}$ wat impliseer dat/ *which implies that* $|x-2|<\frac{\varepsilon}{3}$.

Dit is wat vereis word deur die linkerkant van die bewering/*That is what is required for the left side of the statement*.

Dus / *So*: $\lim\limits_{x\to 2}(3x-5)=1$

6 Inleiding tot funksie-analise/ *Introduction to function analysis*

Voorkennis uit Leereenheid 3 wat vir hierdie leereenheid benodig word:

U behoort reeds uit u skoolkennis of uit die hersiening wat ons in Leereenheid 3 gedoen het, die volgende te kan doen:

1. **Die formele definisie van ‘n funksie as ‘n spesiale relasie kan toepas**
2. **Die definisie- en waardeversameling van ‘n funksie kan identifiseer**
3. **Die inverse van ‘n gegewe funksie kan bepaal**
4. **Bewerkings met funksies kan uitvoer**

Blaai gerus terug indien nodig.

Previously acquired knowledge from Study Unit 3 which is required for this study unit:

From the knowledge you acquired at school or from the revision we did in Study Unit 3, you should already be able to do the following:

1. ***Apply the formal definition of a function as a special relation***
2. ***Identify the domain and range of a function***
3. ***Determine the inverse of a given function***
4. ***Perform operations with functions***

You are welcome to page back if needed.

Leerdoelstellings vir hierdie leereenheid

Na afhandeling van hierdie leereenheid moet die student in staat wees om die volgende te doen:

1. **Die afgeleide van 'n funksie te bereken deur gebruik te maak van die definisie van 'n afgeleide in terme van die limiet van die gradiënt van die snylyn deur 'n kromme wanneer die afstand tussen die**

Learning aims for this study unit

Upon completion of this study unit the student must be able to do the following:

1. ***Calculate the derivative of a function by using the definition of a derivative in terms of the limit of the slope of the secant through a curve when the distance between the points of intersection becomes infinitesimal:***

snypunte infinitesimaal word:

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

2. **Te bepaal of 'n funksie differensieerbaar is in 'n punt of nie, deur van die gradiënt van die raaklyn aan 'n kromme gebruik te maak**

3. **Differensiasiereëls toe te pas om die afgeleide van sekere enkelvoudige funksies te bereken**

4. **Die afgeleide van saamgestelde funksies te bereken**

5. **Differensiasie toe te pas om 'n verskeidenheid funksies kan teken en ontleed**

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

2. ***Determine whether or not a function is differentiable in a point or not, by using the slope of the tangent to a curve***

3. ***Apply differentiation rules in order to calculate the derivative of certain simple functions***

4. ***Calculate the derivative of composite functions***

5. ***Apply differentiation in order to sketch and analyze the graphs a variety of functions***

6.1 Die afgeleide van 'n funksie uit eerste beginsels/ *The derivative of a function from first principles*

Die gradiënt van die snylyn deur 'n kromme as die gemiddelde veranderingstempo van 'n funksie

The slope of the secant intersecting a curve as the average rate of change of a function

Gestel 'n funksie is soos volg gedefinieer:

Suppose a function is defined as follows:

$$f(x) = 3x^2 + 45x$$

Beskou twee punte A en B op die kromme van die funksie sodat A by $x = 3$ en B by $x = 6$ geleë is.

Consider two points A and B on the curve of the function such that A is at $x = 3$ and B is at $x = 6$.

1. Bereken die koördinate van A en B.

1. Calculate the coordinates of A and B.

2. Bereken nou die gradiënt van die snylyn AB.

2. Now calculate the slope of the secant AB.

Let op dat die gradiënt van die snylyn AB eintlik meet "hoe vinnig" die funksiewaarde

Note that the slope of the secant AB actually measures "how fast" the function value op hierdie interval $(3;6)$ verander.

changes over the interval $(3;6)$

Omdat die interval lank is, noem ons die gradiënt van die snylyn die "gemiddelde veranderingstempo van die funksiewaardes met betrekking tot die onafhanklike veranderlike".

Because the interval is long, we refer to the slope of the secant as the "average rate of change of the function values with respect to the independent variable".

Dit is wat ons met die simboool $m_{AB} = \dfrac{\Delta y}{\Delta x}$ bedoel.

That is what we mean by the symbol $m_{AB} = \dfrac{\Delta y}{\Delta x}$.

Die gradiënt van die raaklyn aan 'n kromme as die oombliklike veranderingstempo van 'n funksie

The slope of the tangent to a curve as the instantaneous rate of change of a function

Gestel 'n funksie is soos volg gedefinieer:

Suppose a function is defined as follows:

$$f(x) = 3x^2 + 45x$$

Beskou twee punte A en B op die kromme van die funksie sodat A by $x = 3$ en B by 'n punt $x = 3 + h$ regs van A geleë is.

Consider two points A and B on the curve of the function such that A is at $x = 3$ and B is located at a point $x = 3 + h$ to the right of A.

1. Bereken die koördinate van A en B.

1. Calculate the coordinates of A and B.

2. Bereken nou die gradiënt van die snylyn AB in terme van h .

2. Now calculate the slope of the secant AB in terms of h .

3. Gestel nou ons maak die waarde van h baie klein, sodat die punt B geweldig naby aan die punt A kom. Dan sal die snylyn AB al hoe meer soos 'n raaklyn aan die kromme by A begin lyk.

$m_{AB} \to m_{raaklyn\ by\ A}$ as $h \to 0$

Dus kan ons die naderskuif van B na A ten einde die snylyn AB in 'n raaklyn te verander, soos volg formuleer:

$$m_{raaklyn\ by\ A} = \lim_{h \to 0} m_{AB}$$

In terme van ons gewone notasie:

3. Now suppose we make the value of h very small, so that the point B approaches the point A. Then the secant AB will start to resemble a tangent to the curve at A.

$m_{AB} \to m_{\tan gent\ at\ A}$ *if* $h \to 0$

So we may formulate the movement of point B to A in order to change the secant AB into a tangent, as follows:

$$m_{\tan gent\ at\ A} = \lim_{h \to 0} m_{AB}$$

That is, in terms of our usual notation:

$$m_{raaklyn\ by\ A} = \lim_{h\to 0}\frac{\Delta y}{\Delta x}$$
$$= \lim_{h\to 0}\frac{y_B - y_A}{x_B - x_A}$$
$$= \lim_{h\to 0}\frac{f(3+h)-f(3)}{3+h-3}$$

$$m_{\text{tan}gent\ at\ A} = \lim_{h\to 0}\frac{\Delta y}{\Delta x}$$
$$= \lim_{h\to 0}\frac{y_B - y_A}{x_B - x_A}$$
$$= \lim_{h\to 0}\frac{f(3+h)-f(3)}{3+h-3}$$

Let daarop dat u nou net hierbo vir $\frac{f(3+h)-f(3)}{3+h-3}$ in terme van h bereken het. Al wat u nou hoef te toen om die gradiënt van die raaklyn by A te kry, is om die waarde van h infinitesimaal (oneindig klein) te laat word:

Note that you have just calculated the value of $\frac{f(3+h)-f(3)}{3+h-3}$ in terms of h. So, all you need to do now in order to obtain the slope of the tangent at A, is to let the value of h become infinitesimal (infinitely small):

Die waarde wat u pas verkry het, gee die gradiënt van die raaklyn aan die kromme by die punt A, dit is waar $x=3$.

The value which you obtained gives the slope of the tangent to the curve at the point A, that is where $x=3$.

Aangesien die interval waaroor die gradiënt van die raaklyn bereken is, dit is die grootte van h, infinitesimaal is meet die gradiënt van die raaklyn dus die oombliklike veranderingstempo van die funksiewaarde met betrekking tot die onafhanklike veranderlike.

Because the interval over which we slope of the tangent has been calculated, that is the magnitude of h, is infinitesimal we should note that the slope of the tangent measures the instantaneous rate of change of the function value with respect to the independent variable.

Ons dui die oombliklike veranderingstempo, dit is die gradiënt van die raaklyn, aan met die simbool $\frac{dy}{dx}$ en

We indicate the instantaneous rate of change, that is the slope of the tangent, with the symbol $\frac{dy}{dx}$ and we refer to it as the noem dit die afgeleide van die funksie. *derivative of the function.*

Voorbeeld ***Example***

Bepaal die afgeleide van die funksie $f(t)=4t^3-2t^2$ uit eerste beginsels

Calculate the derivative of the function $f(t)=4t^3-2t^2$ *from first principles*

Oplossing ***Solution***

Beskou punte $A\left(t;\,4t^3-2t^2\right)$ en $B\left(t+h;\,4(t+h)^3-2(t+h)^2\right)$ op die kromme van f.

Consider points $A\left(t;\,4t^3-2t^2\right)$ *and* $B\left(t+h;\,4(t+h)^3-2(t+h)^2\right)$ *on the curve of* f.

Die twee punte is dus :

So the two points are

$$A\left(t;\,4t^3-2t^2\right)$$

$$B\left(t+h;\,4t^3+12ht^2+12h^2t+4h^3-2t^2-4ht-2h^2\right)$$

$$\begin{aligned}
\text{Slope of secant } AB &= \frac{\Delta f}{\Delta t} \\
&= \frac{f_2-f_1}{t_2-t_1} \\
&= \frac{4t^3+12ht^2+12h^2t+4h^3-2t^2-4ht-2h^2-\left(4t^3-2t^2\right)}{t+h-t} \\
&= \frac{4t^3+12ht^2+12h^2t+4h^3-2t^2-4ht-2h^2-4t^3+2t^2}{h} \\
&= \frac{12ht^2+12h^2t+4h^3-4ht-2h^2}{h} \\
&= \frac{h\left(12t^2+12ht+4h^2-4t-2h\right)}{h} \\
&= 12t^2+12ht+4h^2-4t-2h
\end{aligned}$$

Laat B na A beweeg deur h baie klein te maak/ *Let B move to A by making h very small* :

$$\begin{aligned}
\text{Slope of tangent at A} &= \lim_{h\to 0}\left(\text{ Slope of secant } AB\right) \\
&= \lim_{h\to 0}\left(12t^2+12ht+4h^2-4t-2h\right) \\
&= 12t^2+0+0-4t-0 \\
&= 12t^2-4t
\end{aligned}$$

Let daarop dat sommige wiskundiges verkies om die berekening vanuit eerste beginsels effens meer kompak te doen:

Note that some mathematicians prefer to present the calculation of a derivative in a more compact form:

Voorbeeld

Example

Bepaal die afgeleide van die funksie $y = \sqrt{x}$ uit eerste beginsels

Calculate the derivative of the function $y = \sqrt{x}$ from first principles

Oplossing

Solution

$$
\begin{aligned}
\frac{dy}{dx} &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h} \\
&= \lim_{h\to 0}\left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\times 1\right) \\
&= \lim_{h\to 0}\left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\times\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right) \\
&= \lim_{h\to 0}\frac{x+h-x}{h\sqrt{x+h}+\sqrt{x}} \\
&= \lim_{h\to 0}\frac{\cancel{h}}{\cancel{h}\sqrt{x+h}+\sqrt{x}} \\
&= \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} \\
&= \frac{1}{\sqrt{x+0}+\sqrt{x}} \\
\therefore \frac{dy}{dx} &= \frac{1}{2\cdot\sqrt{x}}
\end{aligned}
$$

Oefening 6.1

Bereken die afgeleides van die volgende vanuit eerste beginsels:

Exercise 6.1

Calculate the derivatives of the following functions from first principles

1. $g(t)=\dfrac{3}{\sqrt{t}}$

2. $s(t)=\dfrac{2t-3}{3t+2}$

3. $f(x) = 3x^2 - 3x + 4$

4. $f(x) = \sin x$ **wenk/hint:** $\lim_{h \to 0} \frac{\sin h}{h} = 1$ **en/and** $\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$

5. $f(x)=\cos x$ **wenk/hint:** $\lim_{h\to 0}\frac{\sin h}{h}=1$ **en/and** $\lim_{h\to 0}\frac{\cos h-1}{h}=0$

6. $g(p)=|2p|$

6.2 Differensieerbaarheid/ *Differentiability*

Uit die vorige Leergedeelte volg dat die konsep van 'n afgeleide onlosmaakbaar deel is van die konsep van 'n raaklyn.

Daarom kan ons differensieerbaarheid beskou as 'n eienskap wat beteken:

Indien 'n funksie differensieerbaar is op 'n oop interval, dan is dit by elke punt in daardie oop interval moontlik om 'n unieke raaklyn met 'n reëlwaardige gradiënt aan die funksie te trek.

Dit kom in beginsel daarop neer dat 'n funksie differensieerbaar is waar dit kontinu en glad is.

'n Funksie is dus ondifferensieerbaar waar dit 'n skerp punt vertoon, diskontinu is of vertikaal loop.

From the previous Study Section follows that the concept of a derivative is inseparably part of the concept of a tangent.

Therefore, we may consider differentiability as a property which means:

If a function is differentiable on an open interval, then it is at any point in that open interval possible to draw a unique tangent with real-valued slope to the curve of the function.

In principle this comes down to the fact that a function is differentiable where it is continuous and smooth.

So, 'n function is non-differentiable where it has a shart point, where it is discontinuous or where it runs vertically.

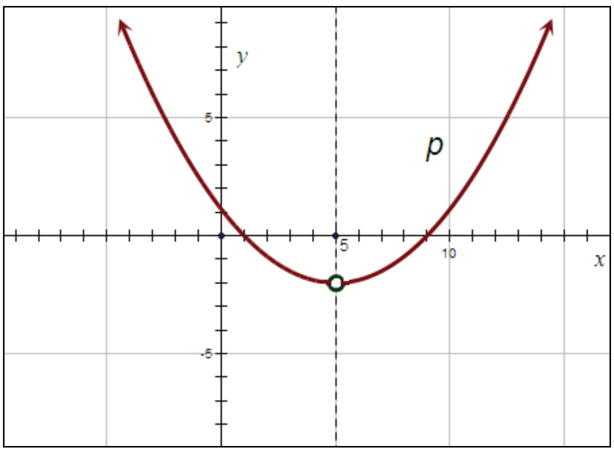
Oefening 6.2

1. Ondersoek al die voorwaardes vir differensieerbaarheid en spesifiseer watter van die voorwaardes verbreek word by elkeen van onderstaande gevalle.

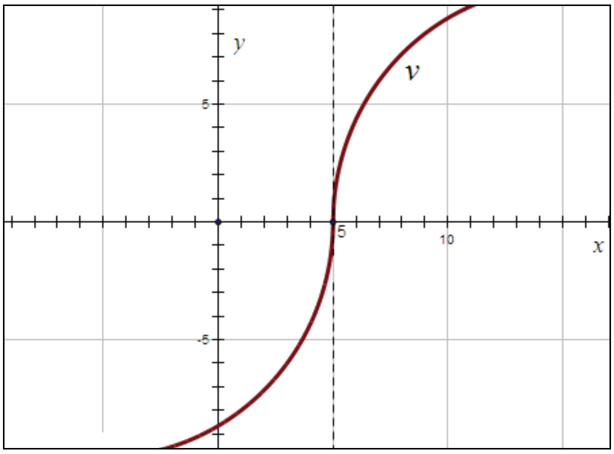
Exercise 6.2

1. Investigate all the conditions for differentiability and specify which of the conditions do not hold at $x=a$ for each of the following cases*:*

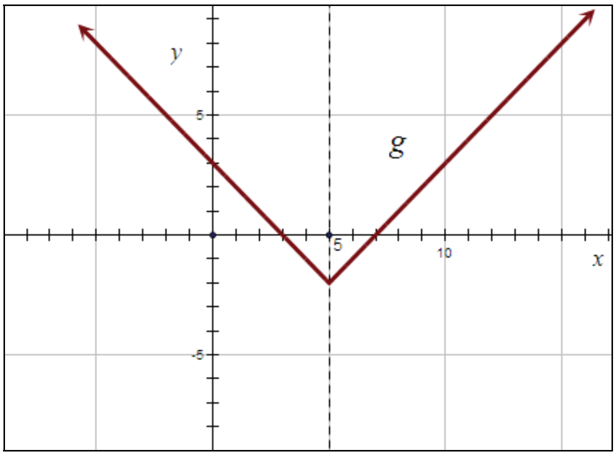
1.1

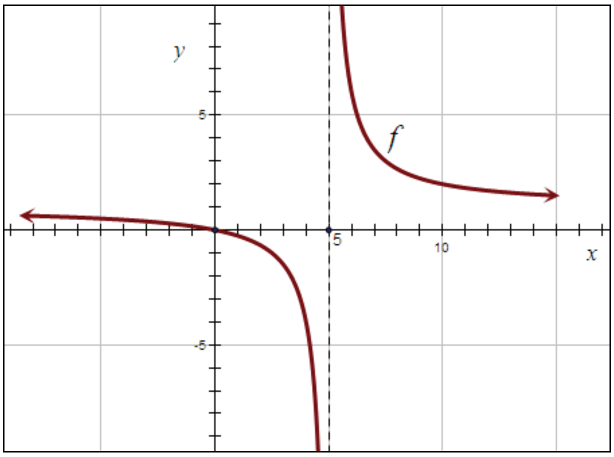


1.2



1.3




y
5
-5
f
5
10
x

6.3 Differensiasiereëls/ *Differentiation rules*

In Leergedeelte 6.1 het ons beleef dat differensiasie vanuit eerste beginsels 'n omslagtige proses is.

In Study Section 6.1 we experienced that differentiation from first principles is a long and tedious process.

Gelukkig word gerieflike differensiasiereëls maklik afgelei deur die definisie van 'n afgeleide op 'n verskeidenheid funksies toe te pas.

Fortunately, convenient differentiation rules may easily be derived by applying the definition of a derivative to a variety of functions.

U sal sommige van hierdie differensiasiereëls nog self bewys; vir vandag aanvaar ons die volgende differensiasiereëls sonder bewys:

You shall prove or derive some of these differentiation rules later in the course of your studies; for today we shall asume their validity without proof:

Konstantes/ *Constants*: D_x (constant) = 0

Optelling/ *Addition*: $D_x[f(x)+g(x)] = D_xf(x)+D_xg(x)$

Aftrekking/ *Subtraction*: $D_x[f(x)-g(x)] = D_xf(x)-D_xg(x)$

Skalaarvermenigvuldiging/ Scalar multiplication: $D_x[cf(x)] = cD_xf(x)$

Magsfunksie/ *Power function* : $D_x x^n = nx^{n-1}$ where n is a rational number.

Produkreël/ *Product rule:* $D_x[f(x)g(x)] = f(x)D_xg(x)+g(x)D_xf(x)$

Kwosiëntreël/ *Quotient rule*: $D_x \dfrac{f(x)}{g(x)} = \dfrac{g(x)\ D_xf(x)-f(x)\ D_xg(x)}{(g(x))^2}$, where $g(x)\neq 0$.

Kettingreël/ *Chain rule:* $\dfrac{dy}{dx}=\dfrac{dy}{dv}\dfrac{dv}{du}\dfrac{du}{dx}$, with $y=f(v)$; $v=g(u)$, $u=h(x)$

Trigonometriese funksies/ *Trigonometric functions*:

$D_x \sin x = \cos x$ $\quad D_x \cos x = -\sin x$ $\quad D_x \tan = \sec^2 x$

$D_x \csc x = -\csc x \cot x$ $\quad D_x \sec x = \sec x \tan x$ $\quad D_x \cot x = -\csc^2 x$

Eksponensiële funksies/ *Exponential functions*:

$D_x e^x = e^x$ $\quad D_x a^x = a^x(\ln a)$ $\quad D_x e^u = e^u D_x u$

Logaritmiese funksies:

$D_x \ln x = \dfrac{1}{x}$ $\quad D_x \log_a x = \dfrac{1}{x}(\log_a e)$ $\quad D_x \ln u = \dfrac{1}{u}D_x u$

Absolute waarde funksie: $D_x|\,x\,| = \dfrac{x}{|x|}$

Laat ons nou ondersoek hoe hulle gebruik word.

Let us now investigate their use.

Oefening 6.3

Exercise 6.2

Bereken die afgeleides in elkeen van die volgende gevalle deur van differensiasiereëls gebruik te maak.

Calculate the derivatives in each of the following cases by making use of differentiation rules:

1. Bepaal die gemiddelde gradiënt van $f(x) = x^2 + 2$ tussen $x = 3$ en $x = 5$/

   Calculate the average slope of $f(x) = x^2 + 2$ *between* $x = 3$ *and* $x = 5$

2. As $f(x) = 3x^2 + 2$, bepaal die helling van raaklyn aan die kromme f by die punt $x = -2$ /

   If $f(x) = 3x^2 + 2$, *determine the slope of the tangent to the curve f at the point* $x = -2$

Differensieer met betrekking tot *x*:/ *Differentiate with respect to x*

(a) $f(x) = 4x^3 + x\sqrt{x} - \dfrac{5}{x^4}$

(b) $f(x) = \sqrt[4]{4x^2 - 6x}$ (skryf in eksponentvorm/ *write in exponential form*)

(c) $f(x) = (2x^2 + 4x)^{100}$

(d) $f(x) = (x^2 + 6x)(x^3 - 6x^2)$ (twee maniere/ *two methods*)

(e) $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$

(f) $f(x) = \dfrac{x^2+3}{5}$

6.4 Saamgestelde funksies en die kettingreël/ *Composite functions and the chain rule*

Oefening 6.4 (hersien 3.5)

Gebruik die kettingreël vir afgeleides, dit is $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$ waar $u=g\left(x\right)$ en bereken die volgende afgeleides.

Exercise 6.4 (revise 3.5)

Use the chain rule for derivatives, that is $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$ where $u=g\left(x\right)$ and calculate the following derivatives.

1.1 $f\left(x\right)=\sqrt{3x-2}$ wenk/ *hint* $\sqrt{a}=a^{\frac{1}{2}}$

1.2 $f\left(x\right)=\frac{1}{\left(x^2-1\right)^2}$ wenk/ *hint* $\frac{1}{a^2}=a^{-2}$

1.3 $f(x)=\left(x^3-x^2+x+1\right)^5$

1.4 $f(x)=\cos 2x$

1.5 $f(x)=\cos^2 x$

6.5 Toepassing van differensiasie/ *Application of differentiation*

Analise is een van die kragtigste gereedskapstukke wat die mens nog ontwikkel het. Alhoewel u nog net van differensiasie kennis dra, kan u alreeds 'n groot verskeidenheid werklikheidsgetroue situasies wat met veranderingstempo te doen het, wiskundig hanteer.

Binnekort sal u ook met integrasie kennis maak.

Die volgende oefening is bedoel om u aan die toepassing van u bestaande kennis bloot te stel.

Analysis (the application of calculus to functions) is one of the most powerful tools ever developed by mankind. Although you have only encountered differentiation calculus at this stage, you are already able to mathematically handle a large variety of problems which involve rates of change.

Soon you will also encounter integration.

The next exercise is meant to expose you to the application of your existing knowledge.

Oefening 6.5 ***Exercise 6.5***

Vraag 1/ ***Question 1***

Die temperatuur van ‘n mengsel in ‘n fabriek verander volgens die funksie $T(t)=-\frac{1}{4}t^2+\frac{5}{2}t+\frac{375}{4}$ vir $0\le t\le 25$ met T in °C en t in minute.

The temperature of a mixture in a factory changes according to the function $T(t)=-\frac{1}{4}t^2+\frac{5}{2}t+\frac{375}{4}$ *for* $0\le t\le 25$ *met* T *in °C en* t *in minutes.*

1.1 Beskou twee punte A en B op die kromme van die funksie T :

Consider two points A and B on the curve of the function T *:*

$$A\left(t;-\frac{1}{4}t^2+\frac{5}{2}t+\frac{375}{4}\right) \text{ en/ and } B\left(t+h;-\frac{1}{4}t^2-\frac{1}{2}th-\frac{1}{4}h^2+\frac{5}{2}t+\frac{5}{2}h+\frac{375}{4}\right)$$

Bereken die gradiënt van snylyn AB.

Calculate the slope of secant AB.

1.2 Bereken vanuit eerste beginsels die gradiënt van die raaklyn aan die kromme by die punt A.

(wenk: laat A $h \to 0$ in die antwoord wat u op vraag 2.1 verkry het)

Calculate from first principles the gradient of the tangent to the curve at the point A.

(hint: Let $h \to 0$ in the answer which you obtained to question 2.1 verkry het)

1.3 Differensieer $T(t) = -\frac{1}{4}t^2 + \frac{5}{2}t + \frac{375}{4}$ op die kort manier.

Differentiate $T(t) = -\frac{1}{4}t^2 + \frac{5}{2}t + \frac{375}{4}$ using the short method.

1.4 Bereken $\left.\frac{dT}{dt}\right|_{t=5\text{ minute}}$ en maak 'n afleiding

Calculate $\left.\frac{dT}{dt}\right|_{t=5\text{ minutes}}$ and draw a conclusion from your answer (what is the uit antwoord (wat impliseer u antwoord)? *implication of your answer)?*

1.5 Bereken die temperatuur op die oomblik wanneer die temperatuurveranderingstempo $-7{,}5\,°\text{C/minuut}$ is.

Calculate the temperature when the rate of temperature change is $-7{,}5\,°\text{C/minute}$.

Vraag 2/ *Question 2*

Gegee: *Given:*

$$f(x) = x^3 - 4x^2 - 11x + 30$$

A en B is die draaipunte van die gegewe funksie.

A and B are the turning points of the given function.

2.1 Skets die grafieke van $f(x)$, $f'(x)$ *en* $f''(x)$ op dieselfde stel asse. Toon die koördinate van die draaipunte (A en B) asook die infleksiepunt aan..

Sketch the graphs of $f(x)$, $f'(x)$ and $f''(x)$ on the same set of axes. Indicate the coordinates of the turning points (A and B) and the point of inflection.

2.2 Wat word bedoel met die begrip "infleksiepunt"?

What is the meaning of the concept "inflection point"?

2.3 Wat is die gemiddelde veranderingstempo van die funksie vanaf A na B?

Calculate the average rate of change of the function from A to B.

2.4 Bereken die vergelyking van die raaklyn aan die kromme by $x = 1$.

Calculate the equation of the tangent to the curve at $x = 1$.

Vraag 3/ *Question 3*

Gegee: *Given:*

$$H(x) = -x^3 + 5x^2 + 8x - 12 = (x-1)(-x^2 + 4x + 12)$$

3.1 Bepaal die koördinate van die kritieke punte. *Find the coordinates of the critical points*

3.2 Pas die eerste-afgeleide-toets en die tweede-afgeleide-toets toe om te toon watter van die kritieke punte is lokale minima of lokale maksima. *Apply the first derivative test and the second derivative test to show whether the critical points are local minima or maxima.*

3.3 Bepaal die koördinate van die punte waar die konkaafheid verander.

Find the coordinate(s) of the points where the concavity changes.

3.4 Pas die tweede-afgeleide-toets toe om te toon dat die konkaafheid verander het.

Apply the second derivative test to show whether the concavity did change

Vraag 4/ *Question 4*

Gegee:

Given:

$$g(x)=\frac{2x+6}{-6x+3}$$

4.1 Bepaal die vertikale en horisontale asimptote deur van limiete gebruik te maak.

Determine the vertical and horizontal asymptotes by utilizing asymptotes.

4.2 Bepaal al die afsnitte met die asse.

Determine all intercepts with axes.

4.3 Skets die grafiek van die funksie. Toon alle inligting aan wat u bepaal het in 4.1 en 4.2.

Sketch the graph of the function. Indicate all the information that you have determined in 4.1 and 4.2.

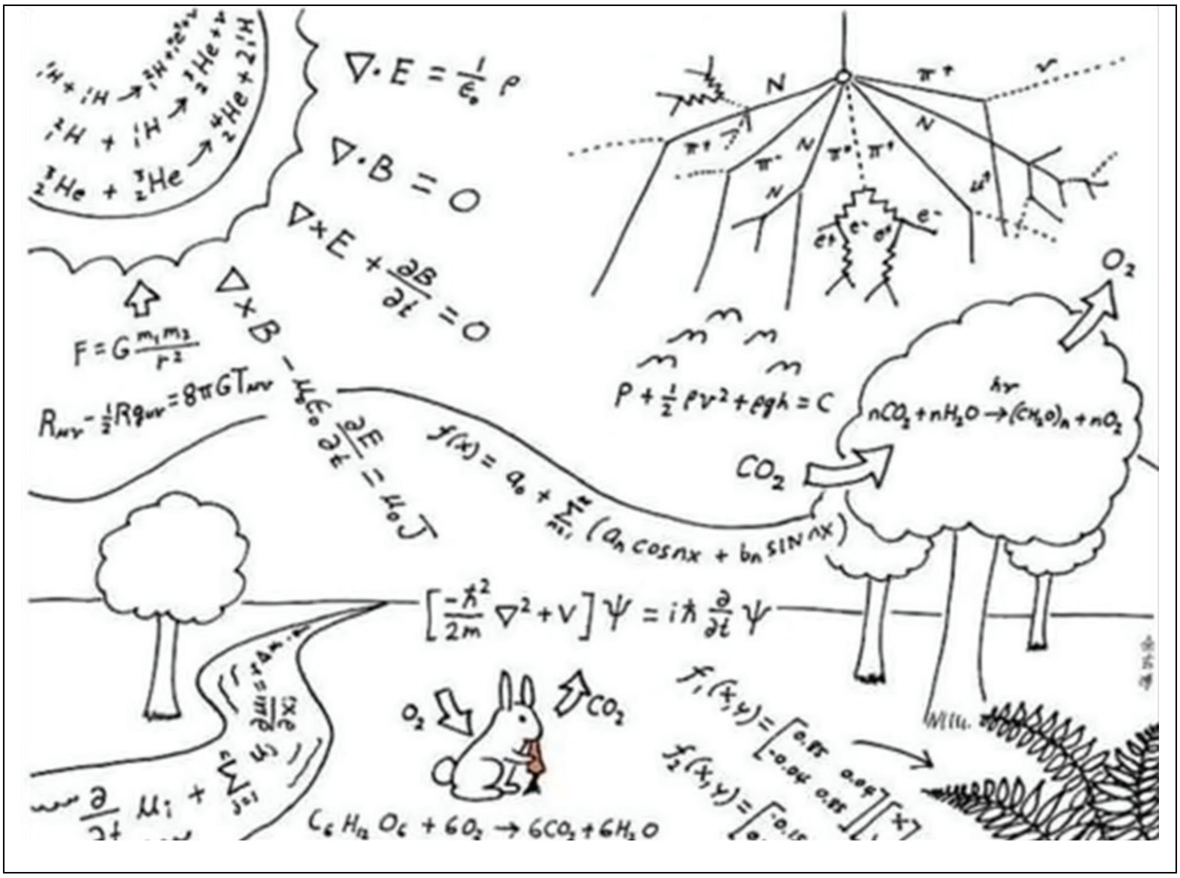
Blaai na die bylae van hierdie boek en voltooi Werkkaart 5.

Turn to the addendum of this book and complete Worksheet 5.

7 Bylaag/ Addendum

Enkele wetenskaplike teorieë in wiskundige simbool-vorm:

A few scientific theories in mathematical symbol form:



NWU

Werkkaart 1

Enige positiewe getal kan as 'n grondtal van die eksponensiële funksie gebruik word. 'n Getal wat dikwels as 'n grondtal in modellering gebruik word is die getal e.

Beskou die uitdrukking $\left(1+\frac{1}{n}\right)^n$.

Ons ondersoek nou wat gebeur as $n \to \infty$, d.w.s. as n baie groot word.

Voltooi onderstaande tabel akkuraat tot 5 desimale plekke:

Worksheet 1

Any positive number can serve as the base of the exponential function. A number that is frequently used as a base in modelling real-life growth is the number e.

Consider the expression $\left(1+\frac{1}{n}\right)^n$.

What will happen if $n \to \infty$*, i.e. if* n *becomes very large.*

Complete the table below correct to 5 decimal places:

n	$\left(1+\frac{1}{n}\right)^n$
1	
10	
100	
1 000	
10 000	
1000 000	
10 000 000	

1. Dit volg dat die uitdrukking $\left(1+\frac{1}{n}\right)^n$

It follows that the expression $\left(1+\frac{1}{n}\right)^n$ *will approach 2,71828…*

na 2,71828… sal streef as $n \to \infty$

as $n \to \infty$ *.*

Bereken nou die waarde van die volgende, ook korrek tot 5 desimale:

Next, compute the value of the following, also accurate to 5 decimal places:

$$\sum_{k=0}^{1} \frac{1}{k!}$$

$$\sum_{k=0}^{3} \frac{1}{k!}$$

$$\sum_{k=0}^{7} \frac{1}{k!}$$

Die getal e word nou gedefineer as:

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \text{ of } e = \sum_{k=0}^{\infty} \frac{1}{k!}$$

The number e *is now defined as:*

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \textit{ or } e = \sum_{k=0}^{\infty} \frac{1}{k!}$$

2. Die funksie $f(x)=e^x$ staan ook bekend as die natuurlike eksponensiële funksie en sy eienskappe is soortgelyk aan die algemene eksponensiële funksie $f(x)=a^x$ met $a>1$.

The function $f(x)=e^x$ is also known as the natural exponential function and its properties are similar to the common exponential function $f(x)=a^x$ with $a>1$.

3. So die natuurlike eksponensiële funksie $f(x)=e^x$ is **stygend vir alle reële waardes van x**.

*So the natural exponential function $f(x)=e^x$ is **increasing for all real values of x**.*

Die natuurlike eksponensiële funksie word gebruik om **kontinue rente**, **bevolkingsgroei** en verskeie ander verskynsels uit die werklike lewe te beskryf.

*The natural exponential function is used to describe **continuous interest**, **population growth** and several other real life phenomena.*

Indien u toegang het tot 'n grafiese sakrekenaar, probeer die volgende ses funksies grafies voorstel en kyk of u die eienskappe wat hierbo genoem is, kan waarneem:

If you have access to a graphical calculator, try to represent the following six functions graphically and see if you can observe the properties listed above:

3.1 $f(x)=\left(1+\frac{1}{x}\right)^x$ (beskou die gedrag waar/ observe the behaviour where $x\to\infty$)

3.2 $g(x)=2^x$

3.3 $h(x)=e^x$

3.4 $m(x)=3^x$

3.5 $r(x)=e^{x-2}-2$

3.6 $u(x) = 2e^{x+1} - 8$

Opgesom:

Daar bestaan in Wiskunde 'n natuurlike grondtal, wat ons met die simbool e aandui. Die waarde van e word in later Wiskunde-kursusse afgelei, maar ons neem die **benaderde waarde** as $e \approx 2{,}718\,28...$

Hierdie grondtal gehoorsaam al die gewone eksponentwette en logaritmiese wette, maar **dit is die gebruik om $\log_e x$ te skryf as $\ln x$.** Ons lees $\ln x$ as "lin ex".

Summarized:

In Mathematics there exists a natural base, which we indicate by the symbol e .*The value of e is derived in later Mathematics courses, but we take its* ***approximate value*** *as*

$e \approx 2{,}718\,28...$

This base obeys the same usual exponential laws and logarithmic laws, but ***it is customary to write*** $\log_e x$ **as** $\ln x$ **.**

We read $\ln x$ as "lin ex".

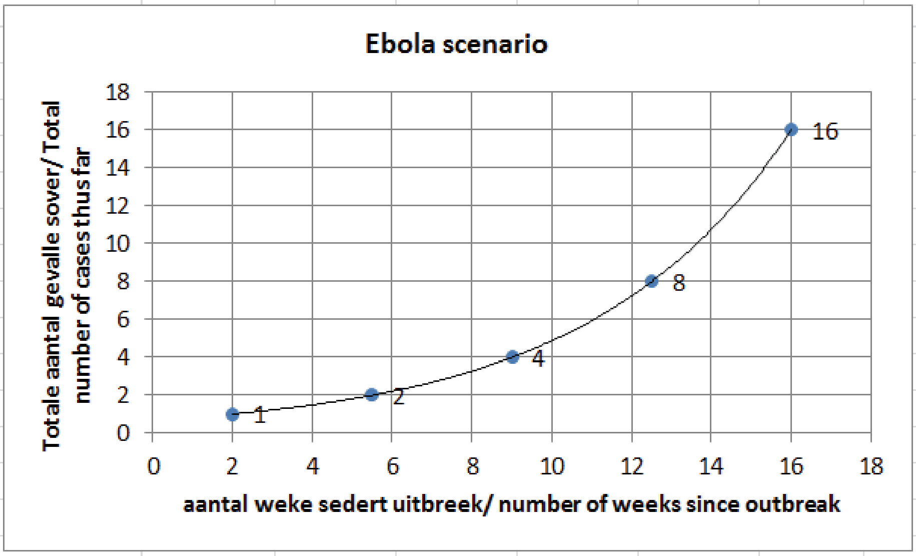
Werkkaart 2 — *Worksheet 2*

1. Aansteeklike siektes — *Contagious diseases*

Die wêreldgesondheidsorganisasie het in November 2014 gemeen dat die totale aantal Ebola-gevalle elke drie tot vier weke verdubbel. 25% tot 90% van die gevalle sterf en toe daar 10 000 gevalle in totaal was, was 5000 van hulle reeds oorlede:

The World Health Organisation reported in November 2014 that the total number of Ebola cases doubles every three to four weeks. 25% to 90% of all cases die and when there were 10 000 cases in total, 5000 of them were deceased:

	31 Des/ Dec 31 2013						31 Okt/ Oct 31 2014
tyd (weke)/ time (weeks)	2	5.5	9	12.5	16		46
totale aantal gevalle/ total number of cases	1	2	4	8	16		10 000



1.1 Bewys dat die totale aantal gevalle nie kwadraties toeneem nie.

(Lê die punte in die gegewe grafiek op 'n parabool?)

Prove that the total number of cases does not increase quadratically.

(Do the points in the given graph lie on a parabola?)

1.2. Wiskundig beskou ons die eksponensiële groei-funksie as die funksie $P(t) = A \cdot 2{,}718^{kt}$.

(Die waarde 2,718 is 'n benaderde waarde vir e, die natuurlike grondtal.)

Aanvaar dat hierdie funksie die vergelyking van die grafiek hierbo is.

Bereken die waardes van A en k.

Mathematically we consider the exponential growth function as the function $P(t) = A \cdot 2{,}718^{kt}$.

(The value 2,718 is an approximate value for e, the natural base.)

Assume that this function is the equation of the graph above.

Calculate the values of A and k.

1.3 Gebruik die laaste punt in die tabel hierbo en bepaal of u antwoorde in 2. korrek is.

(met hoeveel persent verskil u voorspelde waarde vir P met die waarde in die tabel?)

Use the last point in the table above and determine whether or not your answers in 2. are correct.

(with what percentage does your predicted value for P differ from the value in the table?)

1.4 Bepaal die maand en jaar wanneer die waarde van P teoreties 7 miljard (huidige wêreldbevolking) sou wees.

Toets u antwoord.

Determine the month and year when the value of P would theoretically have reached 7 milliard (present global population).

Check your answer.

Opmerking:

Remark:

Die verspreiding van die virus is gelukkig teen die einde van 2014 skynbaar onder beheer gebring. Teen 7 Januarie was daar 20 747 totale gevalle waarvan 8 235 oorlede is.

At the end of 2014 the spread of the virus seemed to be brought under control. By January 7 there was a total of 20 747 of which 8 235 were fatal.

Hieronder is 'n baie meer verfynde model vir die verloop van die siekte:

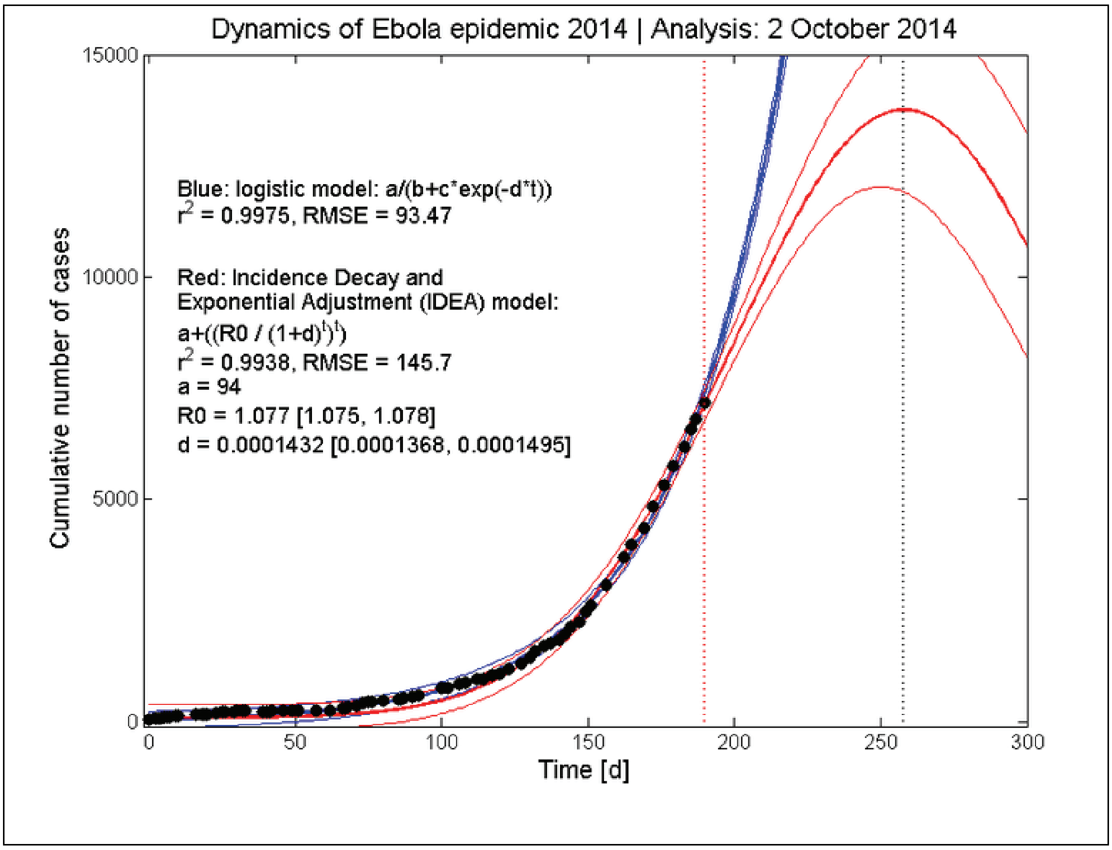
Below is a much more refined model for the procession of the disease:

(Bron:

http://motherboard.vice.com/read/this-math-model-is-predicting-the-ebola-outbreak-with-incredible-accuracy)

(Source:

http://motherboard.vice.com/read/this-math-model-is-predicting-the-ebola-outbreak-with-incredible-accuracy)



2. Radioaktiewe Verval / *Radioactive decay*

Die half-leeftyd van 'n sekere isotoop van die chemiese element Polonium is 138,376 dae (dit neem dus vir 'n sekere massa van $^{210}_{84}Po$ presies 138,376 dae voordat die helfte daarvan in die lood-isotoop $^{206}_{82}Pb$ verander het).

Die vergelyking vir m, die massa $^{210}_{84}Po$ in kg op enige tydstip t (in dae) word gegee deur die vergelyking $m = m_0 e^{-kt}$ waar m_0 die aanvanklike massa Polonium is. Die konstante k word die vervalkonstante genoem.

The half-life of a certain isotope of the chemical element Polonium is 138,376 days (so, it takes 138,376 days for a certain mass of $^{210}_{84}Po$ until half of it has changed into the lead isotope $^{206}_{82}Pb$).

The equation for m, the mass of $^{210}_{84}Po$ in kg at any time t (in days) is given by the equation $m = m_0 e^{-kt}$ where m_0 is the initial mass of Polonium. The constant k is called the decay constant.

2.1 Bereken die waarde van k.

Calculate the value of k.

2.2 Die gemiddelde leeftyd van ${}^{210}_{84}Po$ word in die literatuur aangedui as 200 dae.

As die gemiddelde leeftyd van 'n isotoop gedefinieer word as die resiprook van die vervalkonstante, toets of u antwoord in 2.1 korrek is.

The mean lifetime of ${}^{210}_{84}Po$ is indicated in the literature as 200 days.

If the mean liftetime of an isotope is defined as the reciprocal of the decay constant, check whether your answer in 2.1 is correct or not.

2.3 Bereken hoeveel van 'n 3 kg monster ${}^{210}_{84}Po$ na presies 2 jaar in ${}^{206}_{82}Pb$ verval het.

Toets u antwoord.

Calculate how much of a 3 kg sample of ${}^{210}_{84}Po$ would after exactly 2 years have decayed into ${}^{206}_{82}Pb$.

Check your answer.

2.4 Bereken die aantal dae wat 'n 2 kg monster $^{210}_{84}Po$ sal neem om in 1,75 kg $^{206}_{82}Pb$ te verval.

Calculate the number of days that a 2 kg sample of $^{210}_{84}Po$ *would take to decay into 1,75 kg of* $^{206}_{82}Pb$.

Werkkaart 3

Worksheet 3

1. Die wiskunde van vraag en aanbod

The mathematics of demand and supply

Gestel die vraag na 'n handelaar se produk word gegee deur die funksie $q = 8000 - 40p$, met q die aantal eenhede per week wat deur verbruikers aangevra word as die prys per eenheid p rand is. Die kostefunksie vir die vervaardiging van die produk word gegee deur $C = 100000 + 20q$.

The demand for a merchant's product is given by the function $q = 8000 - 40p$*, with* q *the number of units per week demanded by the consumers if the price per unit is* p *rand. The cost function for the manufacturing of the product is given by* $C = 100000 + 20q$.

1.1 Bepaal die prys en hoeveelheid by die gelykbreekpunte, d.w.s. die punt(e) waar geen wins gemaak word nie.

Determine the price and quantity at the break even points, i.e. the point(s) where no profit is made.

1.2 Bepaal die produksievlak wat 'n maksimum inkomste sal lewer.

Determine the level of production which will yield a maximum revenue

1.3 Bepaal die prys per eenheid wat 'n maksimum inkomste sal lewer.

Determine the price per unit which will yield a maximum revenue.

1.4 Wat is die maksimum inkomste?

Wat is the maximum revenue?

1.5 Bepaal die produksievlak wat 'n maksimum sal wins lewer.

Determine the level of production which will yield a maximum profit.

1.6 Bepaal die prys per eenheid wat 'n maksimum sal wins lewer.

Determine the price per unit which will yield a maximum profit.

1.7 Wat is die maksimum wins?

What is the maximum profit?

2 Finansiële risiko

Financial risk

'n Ekonoom moet 'n moontlike sakegeleentheid ontleed. Die persoon stel voor dat die verwagte wins (P) in rand as 'n funksie van die tyd (t) in maande gegee word deur:

An economist must analyse a prospective business opportunity. The person proposes that the expected profit (P) in rands as a function of time (t) in months is given by:

$$P = 10000 + 30\ t^2 - \tfrac{2}{3}\ t^3$$

2.1 In watter maand(e), indien enige, sal die tempo van groei in wins nul wees?

In which month(s), if any, will the rate of growth in profit be zero?

2.2 In watter maand sal die tempo van groei in wins die vinnigste wees?

During which month will the rate of growth in profit be the fastest?

2.3 Wat is die hoogste groeikoers wat die onderneming sal bereik?

What is the highest rate of growth that the business will achieve?

2.4 Wat is die hoogste wins wat die onderneming vir enige maand sal bereik?

What is the highest profit for any month that the business will achieve?

Werkkaart 4

Worksheet 4

1. Toegepaste trigonometrie-probleme

Applied trigonometric problems

1.1 'n Driehoekige parkeerterrein het afmetings $\hat{B}$ = 29,5°; a = 254 cm; b = 195 cm. Bepaal die oppervlakte van die parkeerterrein indien $\hat{C}$ 'n skerphoek is.

A triangular parking lot has dimensions $\hat{B}$ = 29,5°; a = 254 cm; b = 195 cm. Determine the area of this triangle if $\hat{C}$ is an acute angle.

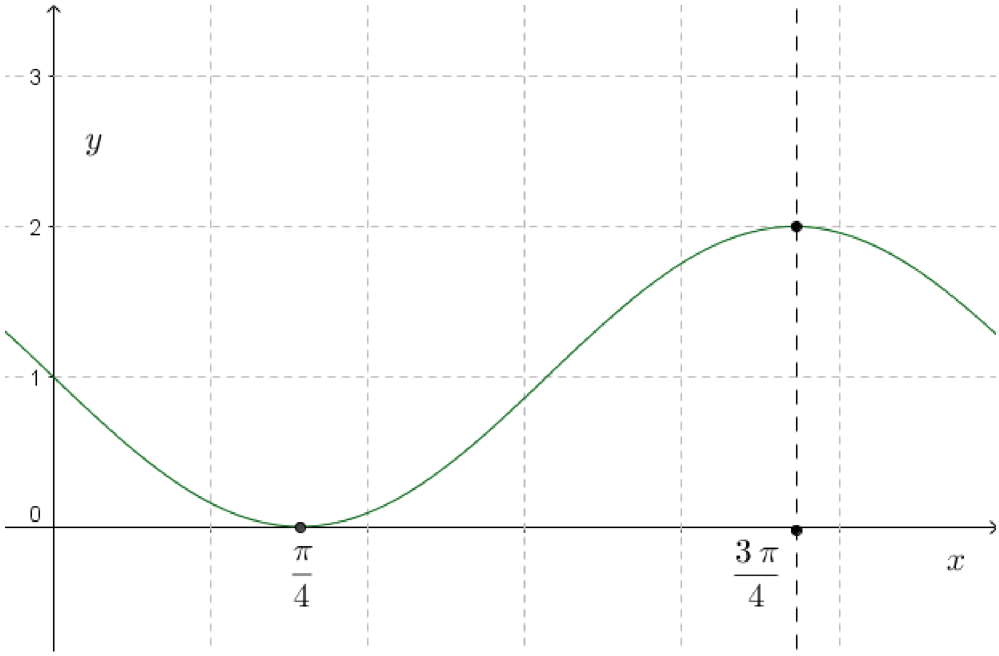
1.2.1 'n Seemyl word gedefinieer as die afstand langs die boog van die oppervlak van die aarde wat 'n hoek van 1 minuut onderspan (1 min = 1/60 grade). Bepaal die waarde van 'n seemyl indien die radius van die aarde 3960 myl is.

A nautical mile is defined as the distance along an arc on the surface of the earth that subtends a central angle of 1 minute (1 min = 1/60 degrees). Find the value of a nautical mile if the radius of the earth is 3960 miles.

1.2.2	'n Vissersboot verlaat Durban se hawe en vaar na Richardsbaai in 'n rigting N 59^{o} E vir 35 seemyl. Die boot draai dan in 'n rigting S 52^{o} E en vaar vir 22 seemyl. Wat is die direkte afstand tussen die boot en Durban by hierdie punt?	*A shipping vessel leaves Durban harbour on a bearing of N 59^{o} E and sails 35 nautical miles in the direction of Richards Bay. It then turns to a bearing of S 52^{o} E and sails 22 nautical miles in this direction. What is the direct distance between the vessel and Durban at this point*?
1.2.3	'n Storm steek op. In watter rigting moet die skip seil om met die kortste roete terug te kom by Durban?	*A storm is brewing. On which bearing must the ship sail to get back to Durban using the shortest route?*

2. Periodiese verskynsels *Periodic phenomena*

2.1 Beskou onderstaande grafiek: *Consider the graph below:*



2.1.1 Bepaal 'n vergelyking vir die kromme in die vorme... *Determine an equation for the curve in the forms...*

a) $y = a\sin(bx + c) + d$

b) $y = a\sin[\omega(x + p)] + d$

$c)\quad y = a\cos(bx + c) + d$

$d)\quad y = a\cos[\omega(x + p)] + d$

2.1.2 Is die funksie ewe, onewe of nie een van die twee nie?

Is the function even, odd function or neither?

2.1.3 Klassifiseer die verskuifde grafieke as ewe, onewe of nie een van die twee nie:

Classify the shifted graphs as even, odd function or neither:

a) Die grafiek word met 1 eenheid afwaarts geskuif.

The graph is shifted downwards by 1 unit.

b) Die grafiek word met 1 eenheid opwaarts geskuif.

The graph is shifted upwards by 1 unit.

c) Die grafiek word na links geskuif met $\pi/4$ eenhede.

The graph is shifted to the left by $\pi/4$ *units.*

2.2 'n Plastiekhouer dobber op die oseaan in eenvoudige harmoniese beweging. Die afstand bo die seebodem word gegee deur:

A plastic holder floating in the ocean is bobbing in simple harmonic motion. Its displacement above the ocean floor is modelled by:

$$y = 0{,}2\cos 20\pi\, t + 8$$

met *y* in meter en *t* in minute.

with y in metres and t in minutes.

2.2.1 Bepaal die frekwensie van die beweging.

Determine the frequency of the movement.

2.2.2 Bepaal die periode van die beweging.

Determine the period of the movement.

2.2.3 Bepaal die maksimum verplasing van die kurk bo die oseaanbedding.

Determine the maximum displacement of the cork above the ocean floor.

2.2.4 Skets die grafiek van die beweging.

Sketch the graph of the movement.

Werkkaart 5 / *Worksheet 5*

(Optimalisering) / *(Optimizing)*

1 ‘n Oop kartonkrat word vervaardig deur vier identiese vierkante uit die hoeke van ‘n vel karton wat 24 cm lank en 18 cm breed is, te sny en die oorblywende gedeeltes na bo te vou om die wande van die krat te vorm.

An open box is manufactured by removing four identical square sections from the corners of a sheet of cardboard which is 24 cm long and 18 cm wide and then folding the remaining parts upwards to form the sides of the container.

1.1 Toon aan dat die volume van die kartonkrat gegee word deur die vergelyking $V = 4x^3 - 84x^2 + 432x$ vir $x \in [0; 9]$.

Show that the volume of the cardboard box is given by the equation $V = 4x^3 - 84x^2 + 432x$ for $x \in [0; 9]$.

1.2 Skets ‘n netjiese grafiek van die funksie V teenoor x vir $-2 \le x \le 14$.

Toon alle snypunte met asse, asook draaipunte en infleksiepunte, duidelik aan in koördinaatvorm.

Sketch a neat graph of the function V against x for $-2 \le x \le 14$.

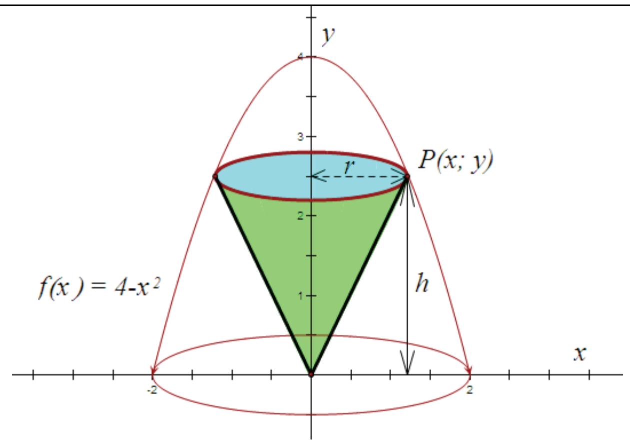
Clearly indicate all intercepts with axes, as well as turning points and points of inflection, in co-ordinate form.

1.3 Wat moet die oppervlakte van elke vierkant wees sodat die volume van die krat 'n maksimum sal wees en wat is die waarde van hierdie maksimum volume?

What should the area of each square be in order for the volume of the box to be a maximum and what is the value of this maximum volume?

2. 'n Regte kegel word gegenereer deur die deel van die parabool $y = 4 - x^2$ tussen die punte $x = 0$ en $x = 2$ om die Y-as te roteer en die kegel in die paraboloïed in te skryf sodat die tophoek op die oorsprong staan en die basis by die punt P aan die parabool raak:

A right cone is generated by revolving the part of the parabola $y = 4 - x^2$ between the points $x = 0$ and $x = 2$ around the Y-axis and inscribing the cone inside the paraboloid so that its vertex is located at the origin and its base touches the parabola at the point P:



2.1 Toon aan dat die volume van die kegel gegee word deur die funksie $V(x) = \frac{4}{3}\pi x^2 - \frac{1}{3}\pi x^4$.

Show that the volume of the cone is given by the function $V(x) = \frac{4}{3}\pi x^2 - \frac{1}{3}\pi x^4$

NWU®

2.2 Skets die kromme van V teenoor x netjies op die grafiekpapier wat voorsien is of in u antwoordskrif.

Neem alle beperkings op die definisieversameling in ag.

Toon alle wortels, draaipunte en infleksiepunte binne die beperkte definisieversameling.

Sketch the curve of V against x neatly on the supplied graph paper or in your answering script.

Take into account all constraints on the domain.

Indicate all roots, turning points and inflection points within the constrainded domain.

2.3 Maak van u berekeninge en resultate uit Vraag 5.1.2 gebruik en bereken die maksimum moontlike volume wat die kegel kan besit.

Employ your calculations and results from Question 5.1.2 and calculate the maximum possible volume which the cone can have.

Neem gerus aan dat die eenhede op die asse cm is.

You are welcome to assume that all units on the axes are cm.