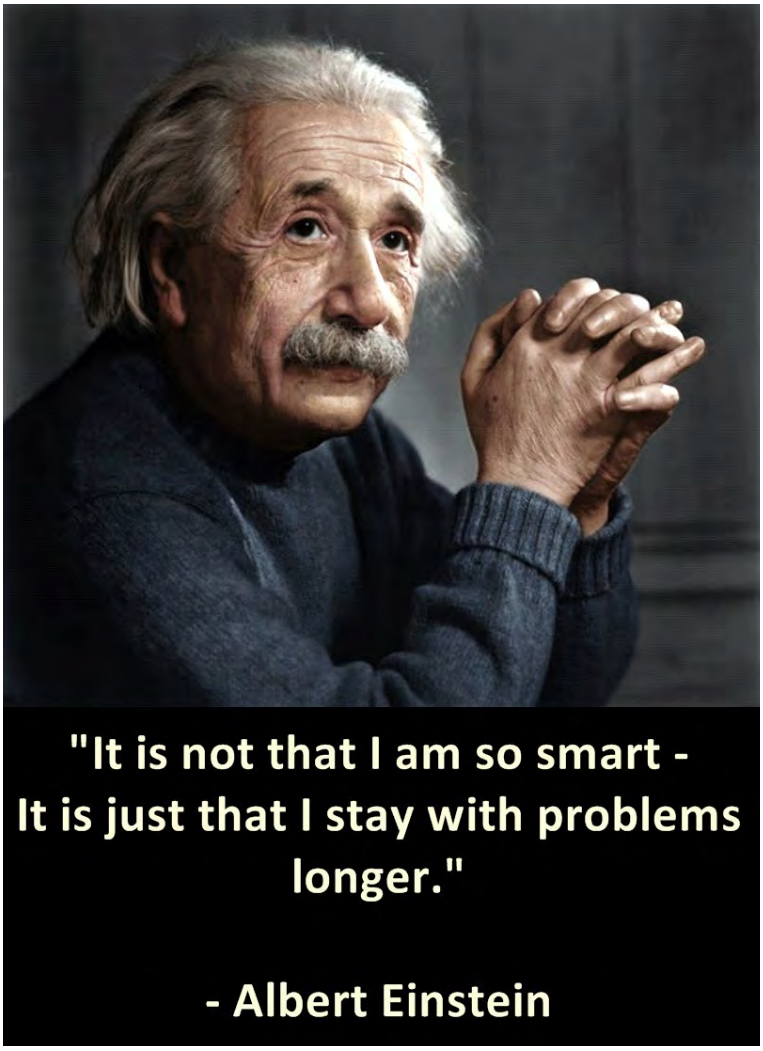
Mathematics and Applied Mathematics

MATHEMATICS INDUCTION GUIDE



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General Information

1 Welcome

We congratulate you as prospective Mathematics student on your decision to commence university studies.

It is a privilege to meet you and to play a part in your preparation for your studies.

We as academic staff hope that your studies will right from the beginning proceed expediently and successfully.

2 Purpose of this course

This course is designed to:

- Activate your previous knowledge regarding Mathematics;
- Jump-start the abstract and logical operation of your brain after the holidays;
- Fill in hazy spots in your mathematical knowledge;
- Equip you with essential knowledge and skills with which you possibly have not become acquainted before;
- Introduce you to the methodology followed during the teaching of university modules;
- Break down the artificial compartments between the different “parts” or “sections” of Mathematics and to reveal the relationships between each “part” and the whole, as well as the relationships between Mathematics and other subject disciplines, and
- Alter your experience of Mathematics as subject discipline to acquire more meaning and life for you than ever before.

3 Study material

In the old days, the study material for any typical mathematics course consisted of one or more text book and class notes, supported by additional material from libraries. The picture is, however, now very different; your study material for any modern Mathematics course will probably consist of a combination of the following:

- Class notes
- One or more text books
- Study material obtained from trustworthy internet sources
- Portable calculator (with or without the ability to generate graphs and tables)
- Interactive computer technology such as specialized programs (software) such as GeoGebra which is installed on desktop or laptop computers
- Applications (“apps”) which are installed on smart phones and tablet computers (for example, GeoGebra for tablets and phones running popular operating systems – Android, iOS, etc.)

You must immediately realize that the days are forever gone when you received all the material which you require from the lecturer or from your text book.

You as mathematician must be able to devise a plan whenever you get stuck with your Mathematics studies – you and your class mates must for example develop the ability to access a search engine (i.e. Google) in order to find information regarding any piece of mathematics. **You do not need a desktop or laptop computer in order to accomplish this; for many years your cell phone has been equipped with the ability to access a search engine and open web pages.**

Web pages such as MathWorks, Wolfram Mathworld and **Wolfram Alpha** are extremely valuable and becomes more user-friendly by the day.

You must also acquire the ability to privately check your reasoning and calculations. In most cases you would need nothing more than a pencil, paper and perhaps a calculator – Mathematics itself as subject discipline functions as one enormous abstract piece of technology.

Your ability to study and learn independently will be one of your most powerful tools in the study of Mathematics.

4 A useful description of what Mathematics is

Mathematics is a large set of abstract ideas which are all connected to one another in a variety of ways. These connections have to do with the meaning of the ideas.

Mathematics is, among other things, a machine or language with which abstract as well as concrete phenomena may be described in an elegant way – a description so elegant that it is not static but capable of being manipulated according to certain conventions and rules in order to shed new light on the original phenomenon. Some thinkers describe Mathematics as “the making of patterns” – sometimes these patterns are visible and measurable and the consequence of human observations; sometimes they are invisible and even immeasurable and can they only be represented symbolically, numerically or graphically by means of **models**.

Mathematical and scientific models are abstract mental representations of real or imaginary patterns. These models may take different forms, for example.

- a set of formulae
- tables filled with numeric data
- a graphical representation
- a computer simulation
- a verbal description

The **Law of Ohm** which was discussed in Grade 9 is an example of a **mathematical model** for the behaviour of current, resistance and potential difference in a simple electrical circuit.

These mathematical models operate according to the principles which are formulated and expressed in the theory of Mathematics.

Mathematical and scientific theory is a very important concept, because the meaning of the word theory is very different in the context of Mathematics and science than its meaning in everyday life.

In Mathematics and science the word “theory” does NOT mean a guess or loose idea or suspicion. A Mathematical or scientific theory is a **rigorous system of consistent ideas and relationships which may successfully be used to explain observations and make predictions**. A theory is only valid as long as it makes sense and has practical value.

Three examples of successful theories:

The **theory of exponents and logarithms** contains the set of concepts, definitions, connections or relationships, formulae and meanings which enable us to calculate, among other things, the final value of a fixed investment at compounding interest. Exponents and logarithms are abstract phenomena, as are the laws, formulae and relationships applicable to them – but the theory works, since we can use it practically to obtain meaningful answers to calculations.

The **theories of gravitation** contain the set of concepts, definitions, connections or relationships, formulae and meanings which enable us to, for example, describe and even predict the behaviour of a free-falling object. Gravity itself is invisible and that is also the case with the universal constant of gravitation and the algebraic rules – the elements of the General Theory of Relativity are even more abstract – and yet, these theories describe, explain and predict the observations we make when objects are subjected to gravity.

The **theories of gravitation** contain the set of concepts, definitions, connections or relationships and meanings which enable us to explain the long-term patterns observed in nature in the studies of species. It explains and describes the fossils observed, as well as the properties and characteristics of the DNA of organisms. Large parts of modern medical science is the consequence of the successful application of the theory of evolution. This theory also explains why a high percentage of the DNA of most modern organisms (including humans) is similar to the DNA of other organisms (even plants and animals).

Theory is therefore extremely necessary in any science, including Mathematics: It provides the framework in which and the mechanisms with which the particular field of study operates. That is why it is fatal to view Mathematics as a mere collection of recipes and rules; Mathematics deals with the patterns and relationships (connections) between abstract concepts – the so-called rules and laws of Mathematics are actually generalizations of the patterns and relationships which we discover when we work with numbers, symbols, operations, etc.

Therefore, your knowledge and skills regarding Mathematics may not exist in neat, separate compartments, like the tools stored on the shelves of a store room.

Rather, it should consist of a network of concepts and skills, where each part of the network is connected to every other part of the network in many ways.

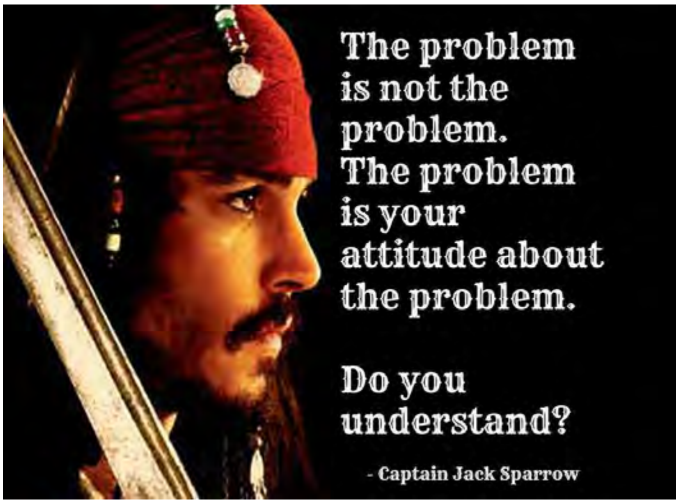
The sub-fields of Mathematics, for example algebra, Euclidean geometry, trigonometry and differentiation, are artificial subdivisions made by humans in order to organize the subject field – **in reality, there does not exist any partition between these sub-fields and each one of them is connected to every one of the others. When we solve a mathematical problem, we typically use ideas which we extract from more than one sub-field**. The proof of a trigonometric identity, for example, requires knowledge of geometry, trigonometry and algebra.

Therefore, your study of Mathematics does not proceed like a house which is built by neatly stacking static bricks, layer upon layer, from bottom to top – rather, it proceeds like a tree, which grows from below to above by means of repeated branching, each branch unique and separate but yet connected to every other part of the tree as a solid but flexible unit.

5 How to master Mathematics

The study of Mathematics requires **focus and time**.

- Be certain at all times that you understand each part of each discussion.
- Mathematics is the science of meaning-making. **If you have to memorize a piece of work because you cannot understand its meaning, then you are no longer busy with Mathematics**.
- If you cannot at first establish a grip on a concept or method, work further and attempt to find an example where the particular concept or method is applied. Work through this example and then return to the discussion that you could not at first understand.
- Attempt to establish where the new piece of work fits in and to identify which previous work is connected to it. **Look for patterns and connections – something is wrong as soon as you consider any new piece of work as a separate isolated part which stands disconnected from the rest of your Mathematics knowledge**.
- **Continually devise plans by which you can check your answers**. That will help to extend and expand your understanding.
- You should become comfortable working with one or more of your classmates. Discuss with one another any new revelations you receive while working through a new piece of work.
- Continually make your own notes and summaries in a form that you understand well.
- **Invest time**. That is your most important resource. Consider again the quote on the cover of this workbook:



1 Logic, algebraic proficiency and exponents

Learning aims for this study unit

Upon completion of this study unit the student must be able to do the following:

1. Understand and apply basic logic principles.
2. Distinguish between the following instructions / action words: Simplify, factorize, solve, differentiate, prove the following identity.
3. Distinguish between the following concepts: Expression, term, numerator, denominator, factor, solution, equation.
4. Correctly apply the properties of real numbers and their operations.
5. Define the absolute value of a number and utilize its properties in calculations and reasoning.
6. Use the properties of exponents in order to simplify expressions.
7. Solve suitable exponential equations by exploiting the properties of exponents.

1.1 Logic

Consider the following statement that you might be familiar with.

If a and b are the side lengths of a right-angle triangle and c the side length of the hypotenuse, then $a^2 + b^2 = c^2$.

This theorem takes the form of a so-called "if-then" statement, or implication. It is important in mathematics that we understand exactly what we mean when we make these kinds of statements. When exactly will this statement be true or false and how can we prove or falsify such a statement. In mathematics we should only accept statements like the one above if there is a formal mathematical proof. To help understand this we will introduce you to the basics of logic.

Whenever you prove a theorem or do a calculation, your work should follow logically. Logic is the study of valid reasoning.

We will need to understand a few concepts. A **sentence** will refer to any combination of words, which may also include numbers or mathematical symbols. Strictly speaking **True** and **False** will simply be labels we give to certain sentences, but you may interpret them in the intuitive way. This brings us to the first important definition.

A **proposition** is a sentence that is either true or false, but not both.

The following sentences are propositions.

1. Plato was a man.
2. $2 + 2 = 4$.
3. $2 + 3 = 4$.
4. The moon is made of cheese.
5. $5 < 3$.

The following are not propositions.

1. Who was Plato? (A question is not a proposition.)
2. $5 + 3$ (There is no truth value to this expression.)
3. $x = 2$ (Until x is specified, this is neither true nor false).

We can construct more complex propositions from other propositions. One can do so using logical connectives. The connectives you will most often see in mathematics are the following.

Negation	$\neg$	not
Conjunction	$\vee$	or
Disjunction	$\wedge$	and
Implication or conditional	$\Rightarrow$	If … then …
Two-way implication or biconditional	$\Leftrightarrow$	… if and only if …

Example

Let A be the proposition "the door is locked", B the proposition "the door cannot open".

1	It is not the case that the door cannot open, i.e. the door can open.	$\neg B$
2	The door is locked, or the door can open.	$A \vee \neg B$
3	The door is locked, and the door cannot open.	$A \wedge B$
4	If the door is locked, then the door cannot open.	$A \Rightarrow B$
5	The door is locked if and only if the door cannot open.	$A \Leftrightarrow B$

Take some time to think what the difference between each of these would be, and under which circumstances they might be true or false.

Sometimes one would like to make arguments such as if $x + 2 < 5$ then $x < 3$. Neither $x + 2 < 5$ or $x < 3$ are strictly speaking propositions, but they would be if we picked a value for x. The convention is then to write it as $x + 2 < 5 \Rightarrow x < 3$. There is a difference between $\rightarrow$ and $\Rightarrow$, but most mathematicians tend to use them interchangeably.

The meaning of each connective is as follows.

1. $\neg A$ is true if A is false and $\neg A$ is false when A is true.
2. $\mathrm{A} \vee \mathrm{B}$ is only false if both A and B are false, otherwise it is true.
3. $\mathrm{A} \wedge \mathrm{B}$ is only true if both A and B are true.
4. $\mathrm{A} \Rightarrow \mathrm{B}$ is false when A is true but B is false, otherwise it is true.
5. $\mathrm{A} \Leftrightarrow \mathrm{B}$ is true when A and B have the same truth values, i.e. they are both true or both false.

For a proposition that consists of many other propositions and connectives, it is often useful to show its truth values in a truth table. The truth table for propositions A and B with the logical connectives discussed is given below.

A	B	$\neg A$	$A \vee B$	$A \wedge B$	$A \Rightarrow B$	$A \Leftrightarrow B$
True	True	False	True	Ture	True	True
True	False	False	True	False	False	False
False	True	True	True	False	True	False
False	False	True	False	False	True	True

1.2 Instructions, action words and introductory concepts in mathematics

Mathematics as human activity is in actual fact a specific, particular way of investigating both the real world and the abstract world.

Whenever an engineer, economist, natural scientist, teacher or any other person "does mathematics" it simply means that he is approaching a situation in a certain way and that he tries to gain better understanding of that situation by proceeding according to certain creative principles.

It is really terribly artificial to reduce mathematics to "the doing of sums" or even to "the solution of problems". Still, that is often what is required of us in mathematics classes or during mathematical tasks and tests.

So, we must first focus on what kind of mathematical problems we may encounter, so that we can become aware of what we must do in each case in order to "obtain the answer".

In mathematics lectures, during mathematics practice session and during mathematics tests we encounter instructions or action words which indicate what we must do. Examples of such action words are:

- Simplify
- Factorize
- Solve for
- Differentiate (calculate $\frac{dy}{dx}$)
- Integrate (calculate $\int f(x)dx$)
- Prove the identity

In the application of mathematics to other fields of study such as accounting, economics, physics, chemistry, computer science and statistics, the real-life problems which we must solve are often not clearly defined in terms of action words:

- Determine the final value of the investment.
- Compare the demand- and supply-graphs.
- Calculate the maximum altitude of the projectile.
- What is the concentration of the solution after 30 seconds?

Usually, in the application of mathematics to other fields of study we ourselves must figure out which mathematical instructions or action words are implied. This skill by which a person interprets real-life problems in mathematical instruction form, is acquired through experience.

Lastly, we should note that "a problem" in mathematics is actually not something negative which should be avoided – as implied by the word "problem" in everyday language. In mathematics "a problem" is in actual fact an instrument or tool by which we acquire knowledge and insight regarding mathematics itself as well as understanding regarding the situation which gave rise to the problem.

A person learns mathematics by studying problem situations and treating them in various ways. Most of the time, the learning value of a mathematical problem is contained in the processes by which the problem is handled, rather than only in its solution. That is why every mathematics problem that you do, has value – even if you are unable to obtain the "most correct answer".

There are many action words in mathematics which you will encounter during further study.

Always ensure that you know the precise meaning of each action word and that you know the form and appearance of the solution associated with that action word (instruction).

Exercise 1.1

1. Consider each of the following given problems. In each case, you must decide what to do (what the instruction for that question should be). In some cases there may be two or more possible instructions. Then do the calculations(s) which goes with the instruction(s).

1.1. $-2x^2 + 7x - 6$

1.2. $(p-2)(p^2 + 2p + 4)$

1.3. $\frac{(10-5t)(2+t)}{t-3} = 0$

1.4. $30k^3 - 22k^2 - 28k = 0$

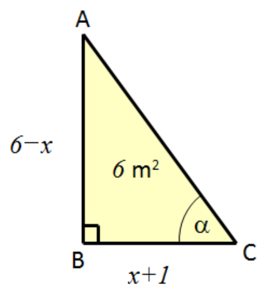
1.5. $f(x) = 6\sqrt{x} - \frac{3}{x} + 5x$

1.6. $p(x) = x^3 + 6x^2 - x + 30$ en $p(3)$.

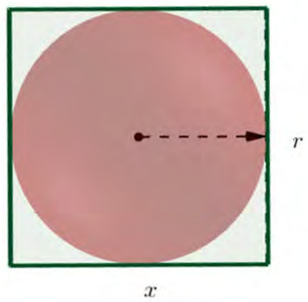
1.7. $x^3 + 6x^2 - x + 30 = 0$ en $x + 2$ is a factor.

1.8. $\sin\theta = \sqrt{1 - \cos^2\theta}$

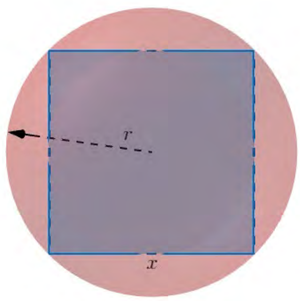
1.9. Given



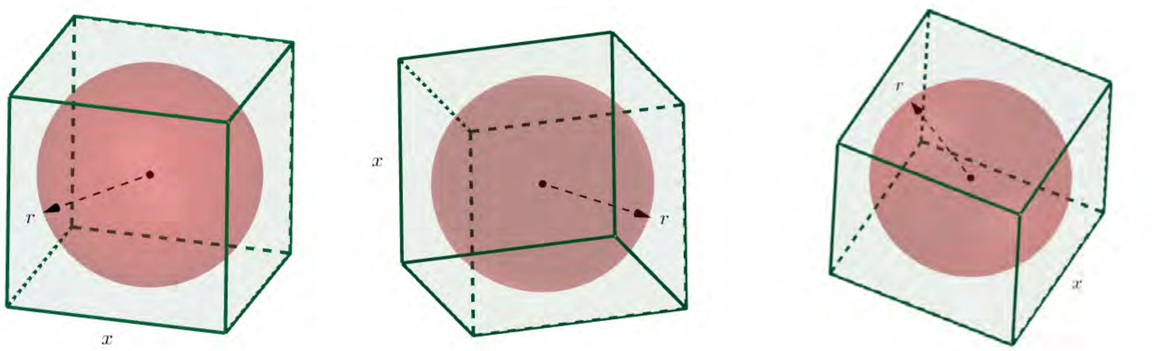
2. Given: A circle that is inscribed inside a square. Prove that the formula for the area of the unshaded region in terms of the radius of the circle is given by $A = r^2(4 - \pi)$.



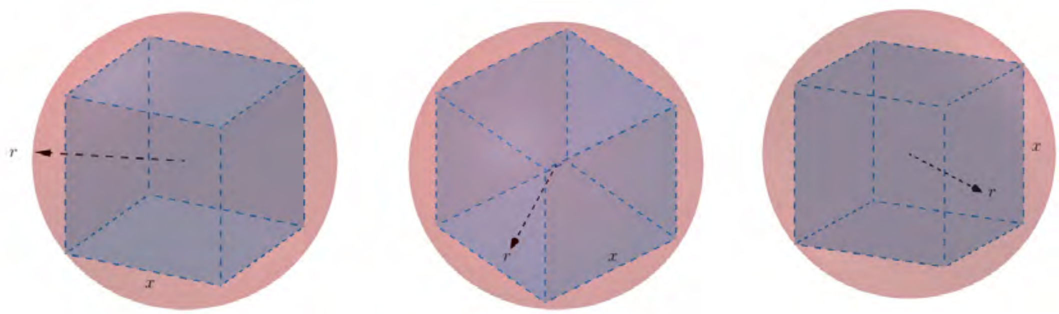
3. Given: A square that is inscribed inside a circle. Prove that the formula for the area of the region that remains if the square is removed, is given in terms of the radius of the circle by the formula $A = r^2(\pi - 2)$.



4. Given: Three views of a sphere that fits perfectly inside a cube. Prove that the volume of the hollow body that is formed by removing the sphere from the cube is given by $V = x^3\left(1 - \frac{\pi}{6}\right)$.



5. Given: Three views of a cube that fits perfectly inside a sphere. Prove that the volume of the hollow body that is formed by removing the cube from the sphere is given by $V = x^3\left(\frac{\sqrt{3}}{2}\pi - 1\right)$.



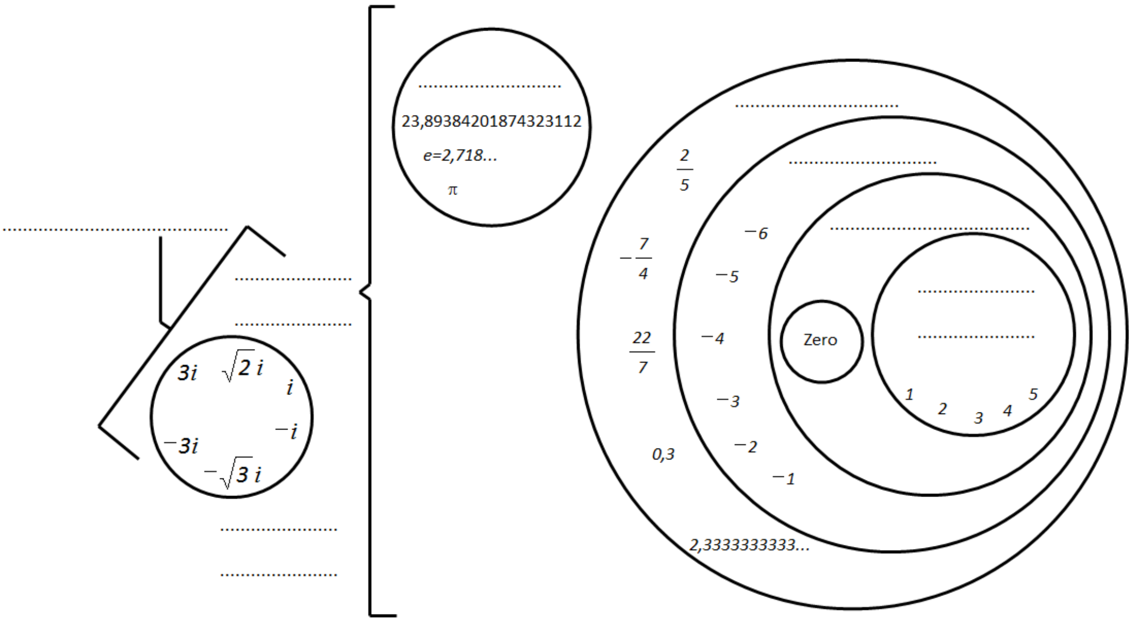
As you can see, each problem is <u>infinitely</u> interesting in its own way, often with different aspects in which we might be interested.

1.3 The real numbers

Number systems

We classify numbers (abstract man-made symbolic and conceptual representations which indicate amount and quantity) as follows according to their properties:

(Supply descriptive names for each of the following sets)



The relationships above may also be expressed in set notation as follows.

$\mathbb{N} = \{1; 2; 3; 4; \ldots\}$	(natuurlike getalle/ natural numbers)
$\mathbb{N}_0 = \{1; 2; 3; 4; \ldots\}$	(whole numbers)
$\mathbb{Z} = \{\ldots; -4; -3; -2; -1; 0; 1; 2; 3; 4; \ldots\}$	(integers)
$\mathbb{Q} = \left\{x \middle| x = \frac{p}{q}, p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0\right\}$	rational numbers)
$\mathbb{I} = \{x \mid x \in \mathbb{R}, x \notin \mathbb{Q}\}$	(irrational numbers)
$\mathbb{R} = \{x \mid x \in \mathbb{Q}$ of / or $x \in \mathbb{I}\}$	(real numbers)

It follows that all the sets above are subsets of the set of real numbers, in other words: $\mathbb{N} \subset \mathbb{N}_0 \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$ and $\mathbb{I} \subset \mathbb{R}$

Further, it follows that $\mathbb{Q} \cup \mathbb{I} = \mathbb{R}$ and $\mathbb{Q} \cap \mathbb{I} = \emptyset$.

What does the symbology above mean in words? You **MUST** know!

Basic properties of the real numbers

Suppose a and b and c are real numbers. Then the following hold:

1. Properties of the number zero (additive identity element)

1.1 $a + 0 =$ ....

1.2 $a \times 0 =$ ....

1.3 $\frac{0}{a} =$ .... with $a \neq 0$

1.4 $\frac{a}{\ldots\ldots\ldots}$ is undefined

1.5 If $ab =$ .... then $a = 0$ or $b = 0$ (...........-product theorem)

1.6 If $\frac{a}{b} = 0$ then $a =$ .... and $b \neq$ ....

2. Operational properties of real numbers

2.1 Closure: $a + b \in \mathbb{R}$ and $ab \in \mathbb{R}$

2.2 Commutativity: $a + b =$ .... and $ab =$ ....

2.3 Associativity: $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$

2.4 Identity: $a + 0 =$ .... and $a \times 1 =$ ....

2.5 Additive inverse: $a + (-a) =$ ....

2.6 Multiplicity inverse $a \times \frac{1}{a} =$ .... where $a \neq 0$

2.7 Distributivity: $a(b \pm c) =$ ....

3. Order of algebraic operations (based on conventions and patterns)

3.1 For repetitive addition and subtraction, we work from left to right.

3.2 For repetitive multiplication and division, we work from left to right.

3.3 For combined operations we cannot simply proceed from left to right, but we should take care to perform addition and subtraction last. Expressions in brackets must be evaluated first – then multiplication and division and lastly addition and subtraction.

3.4 For combined operations the order is as follows:

Priority	Operation	Explanation
1	() parentheses	With sets of parentheses inside other sets of parentheses, proceed from inside out
2	Raising to powers and applying radical operations	Should be treated as a special case of parentheses
3	of	Substitute with a multiply-sign
4	$\times$ and / or $\div$	Same priority
5	$+$ and / or $-$	Same priority

Some mathematicians simplify the scheme above as follows:

Identify all plus and minus signs outside parentheses. Then evaluate all expressions between plus and minus signs and lastly, add and subtract.

The most important matters which you must grasp regarding algebra is:

- Any expression or formula consists of terms – the terms are separated by plus or minus signs (radicals, exponents, multiplication and division does NOT separate terms)
- Whatever stand between plus or minus signs must be regarded as one number. “Algebraic terms” mean “pieces of an expression which is separated by plus or minus signs”.
- Parentheses group everything within together as one numeric value – that is why we always evaluate the contents of a set of parentheses first.
- A horizontal division sign automatically places everything in the numerator and in the denominator in brackets.

4. Illegal or non-permissible operations and real numbers

Simplify the following expression.

$$\frac{3}{7+\sqrt[3]{-343}}+\log_{10}(-10)+\frac{7}{\log_{10}1}-2\sqrt{3+\sqrt[3]{-64}}+\arcsin 2+\arccos(-1.5)-\tan(90^\circ)$$

Use your findings and make list of non-permissible operations. Link each non-permissible operation to the domain of a function.

We must always stay aware of cases like above, where numbers exhibit interesting, unusual behaviour.

Exercise 1.2

1. Determine the value of the following expressions (in other words, **evaluate** the following) without a calculator.

   1.1. $\frac{3(4)}{2}+6(2+3)-\frac{27-7}{7+3}$

   1.2. $4-3|(-2)(3)|+\frac{5^3}{25}-\frac{12}{2+\frac{2}{5}}-\sqrt{5^2-3^2}$

   1.3. $\frac{2\sqrt{169-144}}{5}+\frac{|8-16|}{2^3}-\sqrt{|64-128|}+\frac{100-25}{(10-5)^2}$

   The symbol |a real number| is known as the "absolute value operation". In general, it holds that |a real number| = positive value of that real number. Later more about this.

   1.4.

$$\frac{\frac{7}{\sqrt{144}}-\frac{6^0}{\sqrt[3]{216}}}{\frac{\sqrt[3]{1000}}{(48-3(2)^3}}+\left(\frac{1}{\frac{1}{2}-\frac{\left(\frac{1}{2}\right)}{2}+\frac{5^0}{\sqrt{64}-\frac{2^3}{5-\frac{12-3}{9-\sqrt{\frac{72}{\frac{5}{2}-\frac{1}{2}}}}}}}\right)^{\sqrt{12-8}}-3\left(12-\frac{5}{\left(\frac{1}{2}\right)}\right)^5+\left(\frac{10^2}{10^2+3}\right)^2\cdot\left(1+\frac{3}{10^2}\right)^2$$

2. Determine the value of the following expressions (in other words, **evaluate** the following) without a calculator if possible. Otherwise, explain why not.

   2.1. $\frac{4-\sqrt{16}}{3+2}$

   2.2. $\frac{12-8}{10-\sqrt{100}}$

   2.3. $5-\sqrt{16-25}$

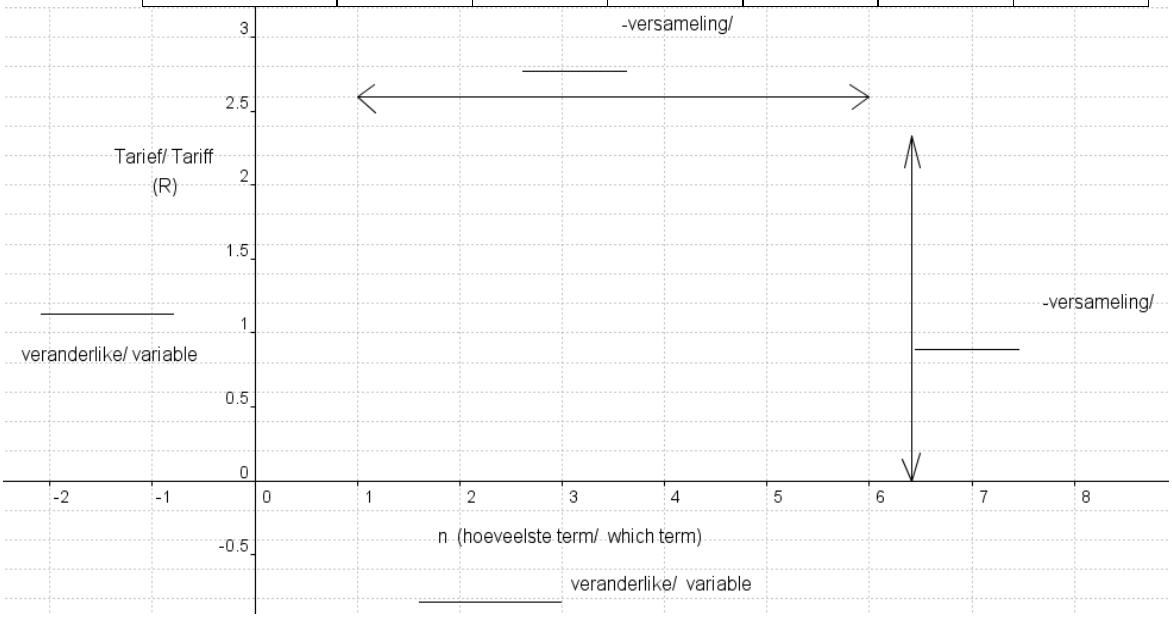
   2.4. $3-2\sqrt{169-25}$

1.4 Exponents

A company intends to apply a tariff increase of 20% per year for the next five years on a service which cost R1.10 in 2016.

Complete the following table and graph.

Year	2016	2017	2018	2019	2020	2021
Term no. (n)	1	2	3	4	5	6
Tariff	1,10					



Do the points lie in a straight line?

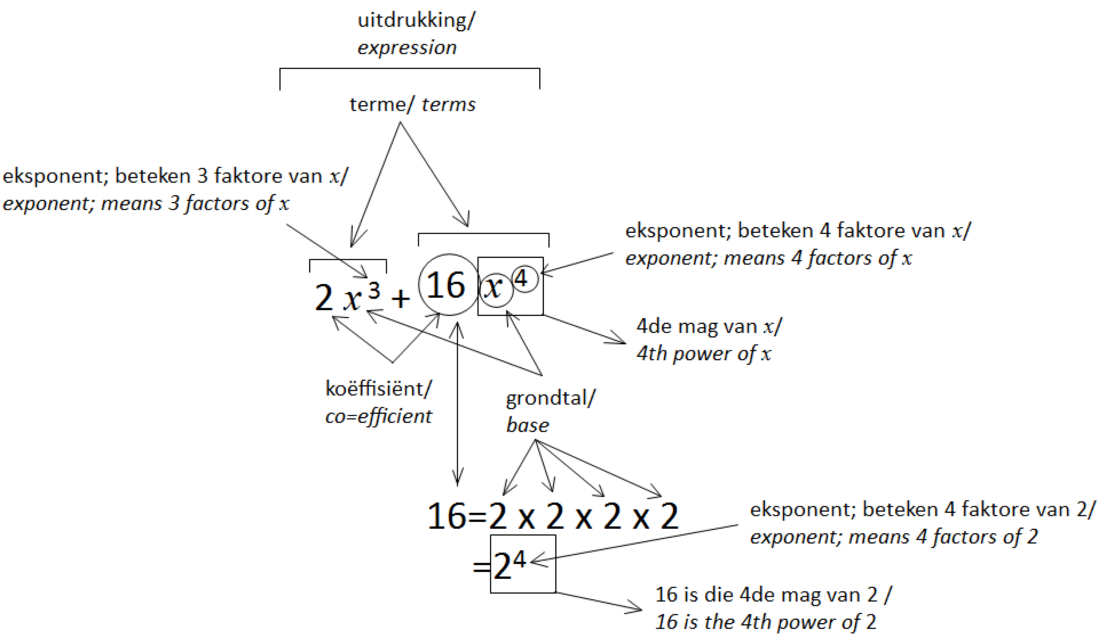
The situation above is an example of **exponential behaviour**; it deals with repetive multiplication by a certain constant value.

Are you able to write down an equation for the graph in terms of n ?

The formula for the general term of a geometric sequence is an example of an **exponential function**.

We define in general $x \times x \times x \times x \times \ldots \times x = x^n$ as n factors of x.

For example, $2x^3 + (2x)^4 = 2x^3 + 16x^4$ then



Use your knowledge and complete the omitted information in the following table.

Properties of exponents

Property:	**Example**	**Beware**
$a^0 = \ldots\ldots\ldots\ldots$	$10^0 = \ldots\ldots\ldots\ldots$	$0^0 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
$a^1 = \ldots\ldots\ldots\ldots$	$5^1 = \ldots\ldots\ldots\ldots$	$0^1 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
$a^m \times a^n = \ldots\ldots\ldots\ldots$	$10^2 \times 10^3 = 100\,000 = 10^{\ldots\ldots}$	$a^m \times a^n \neq a^{mn}$
$\frac{a^m}{a^n} = \ldots\ldots\ldots\ldots$	$\frac{10^6}{10^3} = \frac{1000000}{1000} = 10^{\ldots\ldots}$	$\frac{a^m}{a^n} \neq a^{\frac{m}{n}}$
$\left(\frac{a^m}{b^n}\right)^p = \ldots\ldots\ldots\ldots\ldots\ldots$	$\left(\frac{10^3}{2^2}\right)^4$ $= \left(\frac{1000}{4}\right)^4$ $= 250^4 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ $\frac{10^{12}}{2^8}$ $= \frac{2^{12} \times 5^{12}}{2^8} = 2^4 \times 5^{12}$ $= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$	$\left(\frac{a^m}{b^n}\right)^p \neq \left(\frac{a}{b}\right)^{mp-np}$
$(a)^{-n} = \ldots\ldots\ldots\ldots\ldots\ldots$	$2^{-3} = \ldots\ldots\ldots\ldots\ldots\ldots$ $\left(\frac{3}{2}\right)^{-4} = \ldots\ldots\ldots\ldots\ldots\ldots$	$(a)^{-n} \neq -a^n$ $(a)^{-n} \neq a^{\frac{1}{n}}$
$\sqrt[n]{a^m} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$	$\sqrt{x} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$ $\sqrt[3]{y^6} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$	$\sqrt[n]{a^m} \neq a^{\frac{n}{m}}$ $\sqrt[n]{a^m} \neq a^{m-n}$
$\sqrt[n]{ab} = \ldots\ldots \times \ldots\ldots$	$\sqrt{64t^8} = \sqrt{\ldots\ldots} \times \sqrt{\ldots\ldots}$ $=$	$\sqrt[n]{a+b} \neq \sqrt[n]{a} + \sqrt[n]{b}$ $\sqrt{x^2 \pm y^2} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

The principles above may seem simple and unspectacular, but the errors committed in their application **challenge the imagination.**

Exercise 1.3

1. Simplify without using a calculator.

   1.1. $4(a+b)^4 - 4(a \times b)^2$

   1.2. $\left(\frac{3}{2}x^2\right)\left(\frac{3}{2}x^2\right)\left(\frac{3}{2}x^2\right) - \left(\frac{3}{2}x^2\right)^3 - \left(\frac{3}{2}x^2 + \frac{3}{2}x^2 + \frac{3}{2}x^2\right)^3$

   1.3. $[(z^2)^2]^{-1} \cdot (z^{-1})^3$

   1.4. $(p-m)(p^2 + pm + m^2)$

   1.5. $(2x+3y)(4x^2 - 6xy + 9y^2)$

1.6. $(a-b)^3$

1.7. $\frac{t^3+t^2}{t^5}$

1.8. $\frac{r^3+r^3}{2r^2+3r^2}$

1.9. $\frac{3^{\alpha}.9^{\alpha+1}}{27^{\alpha+2}}$

1.10. $\frac{5^{\beta+2}+5^{\beta+1}}{5^{\beta+2}-5^{\beta+1}}$

2. In Matric you encountered the differentiation rule

$$\frac{d}{dx}(ax^n) = an \cdot x^{n-1}$$

for example,

$$\frac{d}{dx}\left(3x^4 - 5x^2 + \frac{1}{2}x - 9\right) = 12x^3 - 10x + \frac{1}{2}.$$

Now calculate $g'(t)$ if …

2.1. $g(t) = 3\sqrt{t} + \frac{2}{5t}$

2.2. $g(t) = 4\sqrt[3]{t^2} - \frac{5}{3t^2} + \frac{2}{3\sqrt{t}} + \pi$

3. Simplify

$$\frac{4^x \times 5^x}{20} + \frac{10^{3y}}{(2^y \times 5^y)^2}$$

1.5 Simple exponential equations

By exploiting the properties of exponents, we may easily solve certain equations where exponents are involved.

Exercise 1.4

1. Calculate the value of the unknown variable without the use of a calculator.

1.1. $9^{y-2} = 27^{1-2y}$

1.2. $4^{3p} - 32^{2p-5} = 0$

1.3. $3 \cdot 10^x - 0.03 = 0$

1.4. $6 \cdot 2^x - 2^x - 1 = 0$. Explain why there is only one solution.

1.5. $2r^5 = -\frac{243}{16}$. How is this equation different from the previous four?

1.6 The natural number e

Any positive number can serve as the base of the exponential function. A number that is frequently used as a base in modelling real-life growth is the number e.

Consider the expression $\left(1+\frac{1}{n}\right)^n$.

What will happen if $n \to \infty$, i.e. if n becomes very large.

Complete the table below correct to 5 decimal places.

n	$\left(1+\frac{1}{n}\right)^n$
1	
10	
100	
1 000	
10 000	
1000 000	
10 000 000	

It follows that the expression $\left(1+\frac{1}{n}\right)^n$ will approach 2,71828… if $n \to \infty$. Next, compute the value of the following, also accurate to 5 decimal places.

$$\sum_{k=0}^{1}\frac{1}{k!},\quad \sum_{k=0}^{3}\frac{1}{k!},\quad \sum_{k=0}^{7}\frac{1}{k!}$$

The number e is defined as

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \text{ of } e = \sum_{k=0}^{\infty}\frac{1}{k!}$$

The function $f(x) = e^x$ is also known as the natural exponential function and its properties are similar to the common exponential function $f(x) = a^x$ with $a > 1$.

The natural exponential function $f(x) = e^x$ is **increasing for all real values of** x.

The natural exponential function is used to describe **continuous interest**, **population growth** and several other real-life phenomena.

In Mathematics there exists a natural base, which we indicate by the symbol e. The value of e is derived in later Mathematics courses, but we take its **approximate value** a $e \approx 2.71828$ ...

This base obeys the same usual exponential laws**.**

Exercise 1.5

1. Use a graphical calculator or graphing software, to represent the following six functions graphically and see if you can observe the properties listed above.

   1.1. $f(x) = \left(1+\frac{1}{x}\right)^x$. Especially investigate the behaviour for $x \to \infty$.

   1.2. $g(x) = 2^x$

   1.3. $h(x) = e^x$

   1.4. $m(x) = 3^x$

   1.5. $r(x) = e^{x-2} - 2$

   1.6. $u(x) = 2e^{x+1} - 8$

2. Simplify without a calculator but use only in the final step a calculator. Use a caculator to determine the approximate value in the last step.

   $$e^2 + 3e^2 + e^2 \cdot e^{-2} - \frac{e^5}{e^3} + e^0$$

2 Logarithms

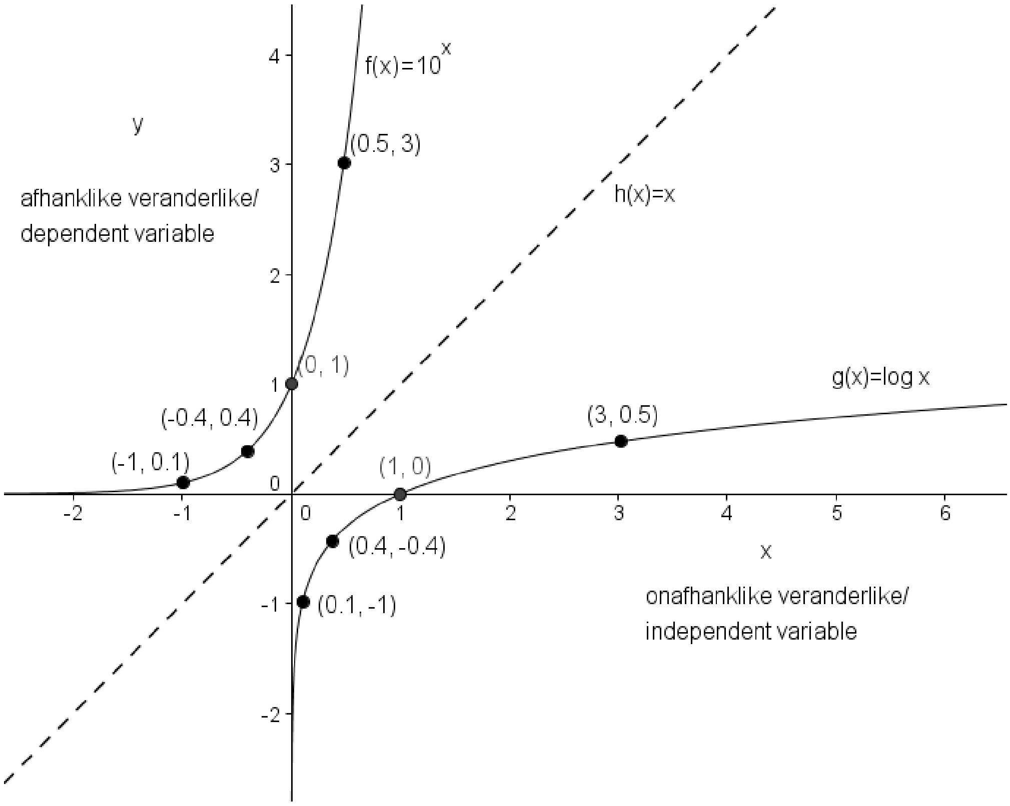
Learning aims for this study unit

Upon completion of this study unit the student must be able to do the following:

1. Use the properties of logarithms in order to simplify expressions
2. Use the properties of logarithms in order to solve logarithmic equations
3. Solve more complicated exponential equations by exploiting the relationships between exponents and logarithms

2.1 Logarithms and exponents

Consider the representation below.



Which observations and conclusions may be made with respect to the exponential operation and the logarithmic operation?

Logarithms basically represent another way to express the same information as that which is given by an exponential equation.

In order to define a logarithm, we employ the theory of inverse functions.

Consider the graph above. If $f(x) = 10^x$ then we may write that $y = 10^x$. In order to obtain the inverse function of $f(x) = 10^x$ we make x the subject by defining that $x = \log_{10} y$. We have swopped the role of the dependant and independent variable. Now $g(y) = \log_{10} y$. Since x and y are abstract variables without any physical meaning, we may swop the symbols for the variables yielding the function $g(x) = \log_{10} x$ as shown in the graph, which is the mirror image of f in the line $h(x) = x$.

We define in general: If $p = a^x$ with $a > 0$ and $a \neq 1$ then $p > 0$ for $x \in \mathbb{R}$ and $x = \log_a p$.

Examples:

$$8 = 2^3 \Leftrightarrow 3 = \log_2 8, \qquad 1000 = 10^3 \Leftrightarrow 3 = \log 1000, \qquad 0.008 = \frac{1}{125} = 5^{-3} \Leftrightarrow -3 = \log_5 0.008$$

Complete the omitted information in the following table.

Properties of logarithms		
Property:	**Example**	**Explanation**
$\log_a 1 = \ldots\ldots\ldots\ldots\ldots$	$\log_3 1 = \ldots\ldots\ldots\ldots\ldots$	"$\log_3 1$"means: $3^{\text{WHAT}} = 1$
$\log_a a = \ldots\ldots\ldots\ldots\ldots$	$\log_e e = \ldots\ldots\ldots\ldots\ldots$	"$\log_7 7$" means: $7^{\text{WHAT}} = 7$
$\log_a \left(\frac{1}{a}\right) = \ldots\ldots\ldots\ldots\ldots$	$\log_5 \left(\frac{1}{25}\right) = \ldots\ldots\ldots\ldots\ldots$	"$\log_7 \frac{1}{343}$"means: $7^{\text{WHAT}} = \frac{1}{343} = \left(\frac{1}{7}\right)^3 = 7^{-3}$
$\log_a (xy) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$	$\log 8 + \log 125$ $= \log\ldots\ldots\ldots\ldots$ $= \ldots\ldots\ldots\ldots$	$a^x \times a^y = a^{\ldots\ldots\ldots}$
$\log_a \left(\frac{x}{y}\right) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$	$\log_5 500 - \log_5 20$ $= \log_5 \ldots\ldots\ldots\ldots$ $= \ldots\ldots\ldots\ldots$	$\frac{a^x}{a^y} = a^{\ldots\ldots\ldots}$
$\log_a \left(x^m\right) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$	$\log_5 625$ $= \log_5 5^4$ $= \ldots\ldots \times \ldots\ldots\ldots\ldots$ $= \ldots\ldots\ldots\ldots\ldots$	$\left(a^m\right)^x = a^{\ldots\ldots\ldots}$
$\log_a b = \frac{\log_c b}{\log_c a}$ For any $c > 0$	$\log_{32} 64$ $= \frac{\log_{\ldots\ldots} 64}{\log_{\ldots\ldots} 32}$ $= \frac{\quad}{\quad}$ $= \ldots\ldots\ldots$	$64 = 2^6$ $\therefore \log_2 64 = 6$ $32 = 2^5$ $\therefore \log_2 32 = 5$ $32^{\frac{6}{5}} = \ldots\ldots\ldots\ldots\ldots$
$\log_a b = \frac{1}{\log_b a}$	$\log_3 27 = \frac{1}{\log_{27} 3}$	$\log_3 27 = \frac{1}{\log_{27} 3}$ $= \frac{1}{\left(\frac{\log_{27} 3}{\log_{27} 27}\right)}$ $= \frac{\log_{27} 27}{\log_{27} 3}$
$a^{\log_a x} = \ldots\ldots\ldots\ldots\ldots\ldots$	$10^{\log_{10} p} = \ldots\ldots\ldots\ldots\ldots\ldots$	The logarithmic operation is the .... of the .... operation

Properties which do NOT hold for logarithms		
Property:	**Example**	**Correct form**
$\log_a(x \pm y) \neq \log_a x \pm \log_b y$	$\log_{10}(1000 \pm 100)$ $\neq \log_{10} 1000 \pm \log_{10} 100$	$\log_a(xy) = \log_a x + \log_a y$
$\log_a(xy) \neq \log_a x \times \log_b y$	$\log_{10}(1000) \neq \log_{10} 100 \times \log_{10} 10$	$\log_a(xy) = \log_a x + \log_a y$
$\log_a \frac{x}{y} \neq \frac{\log_a x}{\log_a y}$	$\log_3 \frac{81}{27} \neq \frac{\log_3 81}{\log_3 27}$	$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
$(\log_a x)^m \neq m \cdot \log_a x$	$(\log_5 25)^3 \neq 3 \cdot \log_5 25$	$\log_a(x^m) = m \cdot \log_a x$ of / *or* $\log_a x^m = m \cdot \log_a x$

It is important to grasp very well the implications of the discussion above, since we require it in many applications.

Exercise 2.1

1. Simplify without using a calculator.

   1.1. $\log_{10} 20 + \log_{10} 50$

   1.2. $\log_6 36 + \log_6 \frac{1}{36} - 3\log_6 1$

   1.3. $\frac{\log 81 - \log 16}{\log 9 - \log 4}$

   1.4. $\log_4 4^{20} + (\log_4 4)^{20} - (\log_4 16) \cdot (\log_4 64)$

   1.5. $(\log_7 49) \log_7 1$

   1.6. $(\log 100) \cdot \log 1000$

   1.7. $3^{\log_3 x}$

   1.8. $\frac{\ln x + 2\ln x}{\ln x^3} - \ln(e^3) + (\ln e)^3 + \ln 1$ (Dit is die gebruik om $\log_e x$ te skryf as $\ln x$. Ons lees $\ln x$ as "lin ex".)

2. Determine the value of the unknown (Solve):

   2.1. $\log_3 x = 4$

   2.2. $\log x + \log(x+3) = 1$

   2.3. $\log(1-2x) - \log(x+2) = \log 1$

   2.4. $\ln x = \ln(2x-1) + 2\ln x$

2.2 Solving more complicated exponential equations using logarithms

Because we defined a logarithm in terms of exponents, we may rewrite more complicated exponential equations to logarithmic form. Such logarithmic equations are then easier to solve than the original exponential equation.

Example

1. Solve for x : $\left(\frac{3}{10}\right)^{5x} = \frac{345}{64}$
2. Solve for t: $3.5 \cdot 0.1^{3t} = 2.5^{t-1}$

Solution

1.
$$\left(\frac{3}{10}\right)^{5x} = \frac{343}{64}$$
$$\therefore 5x = \log_{\frac{3}{10}} \frac{343}{64}$$
$$\therefore 5x = \frac{\log\left(\frac{343}{64}\right)}{\log\left(\frac{3}{10}\right)}$$
$$\therefore x = \frac{\left[\frac{\log\left(\frac{343}{64}\right)}{\log\left(\frac{3}{10}\right)}\right]}{5}$$
$$= -0{,}279$$

2.
$$3{,}5 \cdot 0{,}1^{3t} = 2{,}5^{t-1}$$
$$\therefore \log\left(3{,}5 \cdot 0{,}1^{3t}\right) = \log\left(2{,}5^{t-1}\right)$$
$$\therefore \log(3{,}5) + \log\left(0{,}1^{3t}\right) = \log\left(2{,}5^{t-1}\right)$$
$$\therefore \log(3{,}5) + 3t \cdot \log(0{,}1) = (t-1) \cdot \log(2{,}5)$$
$$\therefore \log(3{,}5) + 3t \cdot \log(0{,}1) = t \cdot \log(2{,}5) - \log(2{,}5)$$
$$\therefore 3t \cdot \log(0{,}1) - t \cdot \log(2{,}5) = -\log(2{,}5) - \log(3{,}5)$$
$$\therefore t\left[3\log(0{,}1) - \log(2{,}5)\right] = -\log(2{,}5) - \log(3{,}5)$$
$$\therefore t = \frac{-\log(2{,}5) - \log(3{,}5)}{3\log(0{,}1) - \log(2{,}5)}$$
$$= 0{,}277$$

Exercise 2.2

1. Solve the following equations.

   1.1. $0.00293 = 3\left(\frac{1}{2}\right)^{t}$

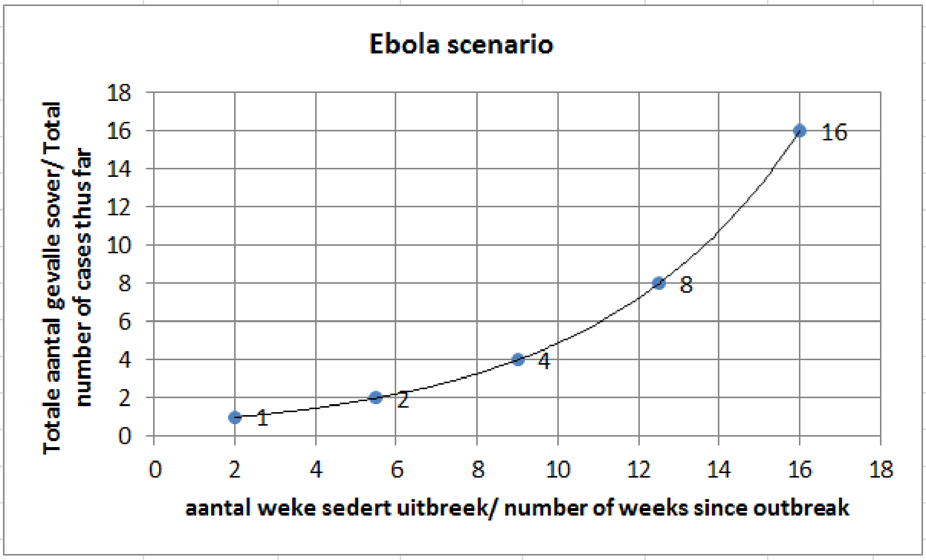
   1.2. $\frac{1}{3}\left(\frac{4}{3}\right)^{n-1} = 2.497180$

   1.3. $4 \cdot 2^{3t} = \frac{1}{2} \cdot 5^{2t-1}$

   1.4. $2 \cdot \left(\frac{1}{3}\right)^{2n} - \frac{1}{2} \cdot 5^{1-3n} = 0$

2. The World Health Organisation reported in November 2014 that the total number of Ebola cases doubles every three to four weeks. 25% to 90% of all cases die and when there were 10 000 cases in total, 5000 of them were deceased:

	31 Des 2013					31 Okt 2014
Tyd (weke)	2	5.5	9	12.5	16	46
Totale aantal gevalle	1	2	4	8	16	10000



2.1. Prove that the total number of cases does not increase quadratically. Do the points in the given graph lie on a parabola?

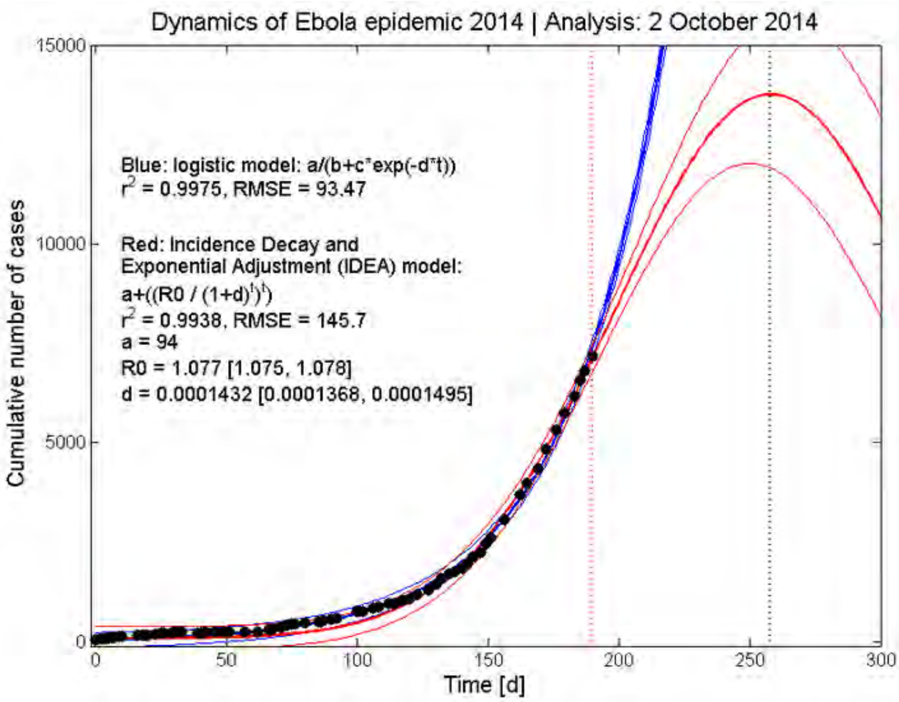
2.2. Mathematically we consider the exponential growth function as the function $P(t) = Ae^{kt}$. Assume that this function is the equation of the graph above. Calculate the values of A and k.

2.3. Use the last point in the table above and determine whether or not your answers in 2.2. are correct. With what percentage does your predicted value for P differ from the value in the table?

2.4. Determine the month and year when the value of P would theoretically have reached 7 billion (10^9) (present global population).

Remark: At the end of 2014 the spread of the virus seemed to be brought under control. By January 7 there was a total of 20 747 of which 8 235 were fatal. Below is a much more refined model for the procession of the disease. (Source:

http://motherboard.vice.com/read/this-math-model-is-predicting-the-ebola-outbreak-with-incredible-accuracy)



3. The half-life of a certain isotope of the chemical element Polonium is 138.376 days (so, it takes 138.376 days for a certain mass of $^{210}_{84}\mathrm{Po}$ until half of it has changed into the lead isotope $^{206}_{82}\mathrm{Pb}$).
The equation for m, the mass of $^{210}_{84}\mathrm{Po}$ in kg at any time t (in days) is given by the equation $m = m_0 e^{-kt}$ where m_0 is the initial mass of Polonium. The constant k is called the decay constant.

3.1. Calculate the value of k.

3.2. The mean lifetime of $^{210}_{84}\mathrm{Po}$ is indicated in the literature as 200 days. If the mean lifetime of an isotope is defined as the reciprocal of the decay constant, check whether your answer in 3.1 is correct or not.

3.3. Calculate how much of a 3 kg sample of $^{210}_{84}\mathrm{Po}$ would after exactly 2 years have decayed into $^{206}_{82}\mathrm{Pb}$.

3.4. Calculate the number of days that a 2 kg sample of $^{210}_{84}\mathrm{Po}$ would take to decay into 1.75 kg of $^{206}_{82}\mathrm{Pb}$.

3 Introduction to functions

Learning aims for this study unit

Upon completion of this study unit the student must be able to do the following:

1. Apply the formal definition of a function as a special relation.
2. Identify the domain and range of a function.
3. Determine the inverse of a given function.
4. Perform operations with functions.

3.1 Definition of a function

Relations

In Mathematics the word relation indicates two sets (groups of numeric values) between which there exists a relationship or connection. For each element taken from the one set a connection can be made to one or more of the elements in the other set. This relationship is usually a certain pattern or **rule**.

This rule may be a verbal instruction, or an algebraic formula, or a table with values, or a graph.

Example of a relation: Complete the pattern $\left(number; \pm\sqrt{number}\right)$ for the numbers $\{0;4;9;16;25\}$.

$$\{(0;\ldots\ldots\ldots);\ (4;\ldots\ldots\ldots\ldots);(9;\ \pm 3);(16;\ldots\ldots\ldots\ldots);(25;\ldots\ldots\ldots\ldots)\}$$

Note that each first element is connected with two other elements.

Functions

A function is **a rule** which **connects** each of the elements in the one set (the domain) with **one and only one element** in the other set (the range).

Example of a function: Complete the pattern $\left(number; number^2\right)$ for the numbers $\{-2;-1;0;1;2\}$.

$$\{(-2;\ldots\ldots\ldots);\ (-1;\ldots\ldots\ldots\ldots);(0;\ \ldots\ldots\ldots\ldots);(1;\ldots\ldots\ldots\ldots);(2;\ldots\ldots\ldots\ldots)\}$$

Note that each first element is connected with only one other element. We call this an **unambiguous relationship.**

3.2 Domain and range

The **domain** is the set of independent elements. The **range** is the set of dependent elements which depends on the elements in the domain.

One-to-one functions

A one-to-one function has the property that each element from the domain is mapped unto a unique, different element of the range.

Example of a one-to-one function: Complete the pattern $\left(number;\ number^3\right)$ for the numbers $\{-2;-1;0;1;2\}$.

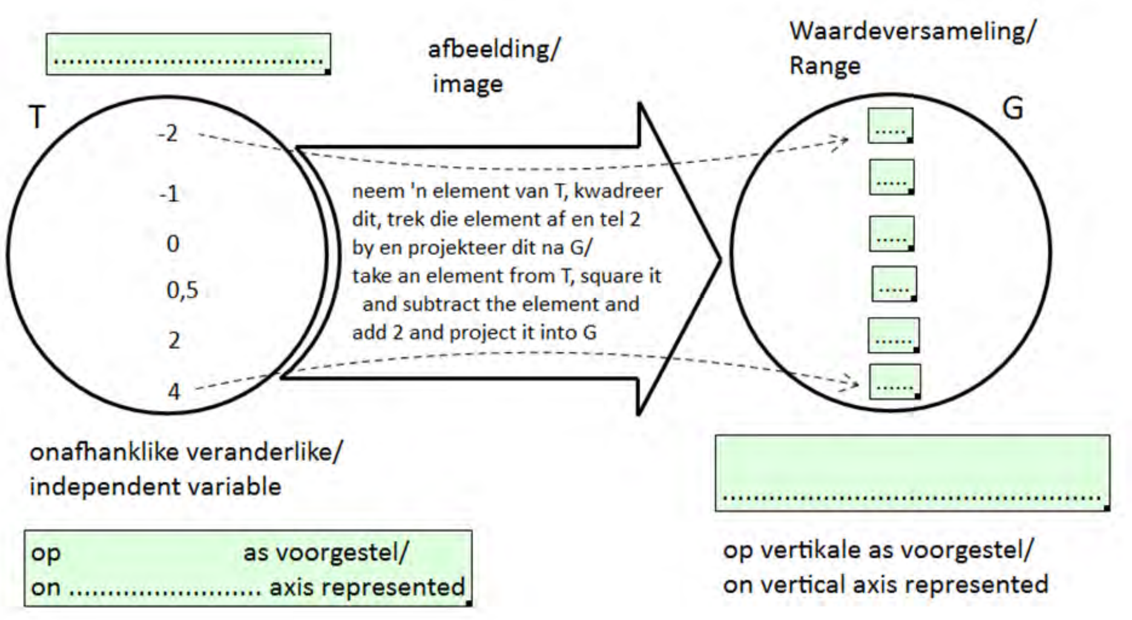
$$\{(-2;\ldots\ldots\ldots);\ (-1;\ldots\ldots\ldots\ldots);(0;\ \ldots\ldots\ldots\ldots);(1;\ldots\ldots\ldots\ldots);(2;\ldots\ldots\ldots\ldots)\}$$

Not one-to-one functions

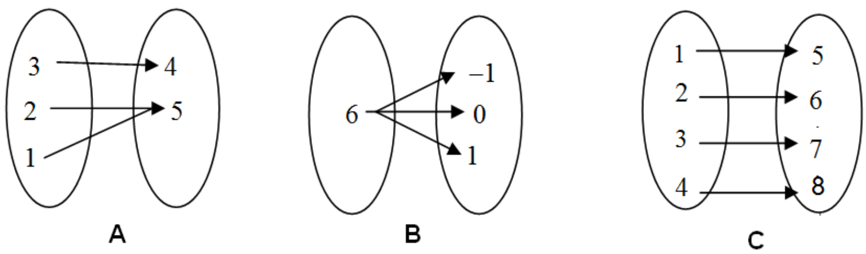
A not one-to-one-function has the property that each element from the domain is not mapped unto a unique, different element of the range. You see this clearly in the example above regarding the squares. Different numbers yield the same squares.

Exercise 3.1

1. Consider the schematic below which shows a function and fill in the omitted concepts.



2. Represent the information in the schematic representation on a set of axis.
3. Perform **interpolation** (draw a smooth curve through the points that you just plotted between and including where $t = -2$ and $t = 4$).
4. Use the given information and write down the equation (formula) of the graph. Hint: at school it would have been something like $y = ax^2 + bx + c$.
5. Describe the shape of the curve: ascending or descending, concavity, minimum and/or maximum turning point(s).
6. Write down the domain T of the function in set notation.
7. Write down the range G of the function in set notation.
8. Now perform **extrapolation** (extend the curve "beyond" the points where $t = -2$ and $t = 4$).
9. Use step 4 from above and calculate the values of $g(-1.5)$, $g\left(\frac{3}{2}\right)$, $g(5)$ and $g(2 + h)$.
10. Indicate on your graph where you would read off the first three answers to question 9.
11. Use step 4 from above and calculate the values of t such that $g(t) = 7$.
12. Indicate on your graph where you would read off the values of your answers to question 11.
13. Consider the representation



13.1. Which of the given cases represent functions?

13.2. Which of the functions above are one-to-one-functions?

14. If $A=\{-1;0;1;2;3\}$ and $B=\{-3;-2;-1;0;1;2;3;...;9;10\}$ write the following functions as sets of ordered pairs.

14.1. $\{(x;y)|y=x; x\in A, y\in B\}$

14.2. $\{(x;y)|y=x^2; x\in A, y\in B\}$

15. Say if each of the functions in Question 14 is a **one-to-one-function** or a **not one-to-one function**.

16. Sketch each of the following functions in rough and write down the domain and range of it.

16.1. $\{(x;y)|y=(x-1)^2-4; x\in\mathbb{R}, y\in\mathbb{R}\}$

16.2. $\{(t;y)|y=-(t-2)^2+2; t\in\mathbb{R}, y\in\mathbb{R}\}$

16.3. $\left\{(z;y)|y=-\frac{8}{z+1}+1; x\in\mathbb{R}, y\in\mathbb{R}\right\}$

16.4. $\left\{(p;r)|r=\sqrt{p-2}; r\in\mathbb{R}, p\in\mathbb{R}\right\}$

16.5. $\{(x;y)|y=e^x; x\in\mathbb{R}, y\in\mathbb{R}\}$

16.6. $\{(x;y)|y=\ln x; x\in\mathbb{R}, y\in\mathbb{R}\}$

Function values

Function values are the dependent variable values which are read off the vertical axis. They are calculated by substituting certain values of the independent variable into the defining equation of the function.

Exercise 3.2

1. If $s(t)=3$ determine $s(0), s(1), s(t+h)$.

2. If $T_n=\frac{7}{2}\left(\frac{1}{2}\right)^{n-1}$ determine T_1 and T_4.

3. If $g(y)=\frac{3}{2y}$ determine $g(0)$, $g(t)$ and $g\left(\frac{2}{3}\right)$.

3.3 Inverse of a function

Inverse functions

Example

If $y=\sin x$ and $x=\frac{\pi}{6}$ then $y=\sin\left(\frac{\pi}{6}\right)$ which yields $y=\frac{1}{2}$.

Suppose, however, we know that $y=\frac{1}{2}$ in $y=\sin x$ but we wish to determine the value of x. Then we go about as follows. $\frac{1}{2}=\sin x$ so $x=\sin^{-1}\left(\frac{1}{2}\right)$ and this yield, among other values, $x=\frac{\pi}{6}$ as solution.

The argument above illustrates the use of an inverse function or inverse function operation.

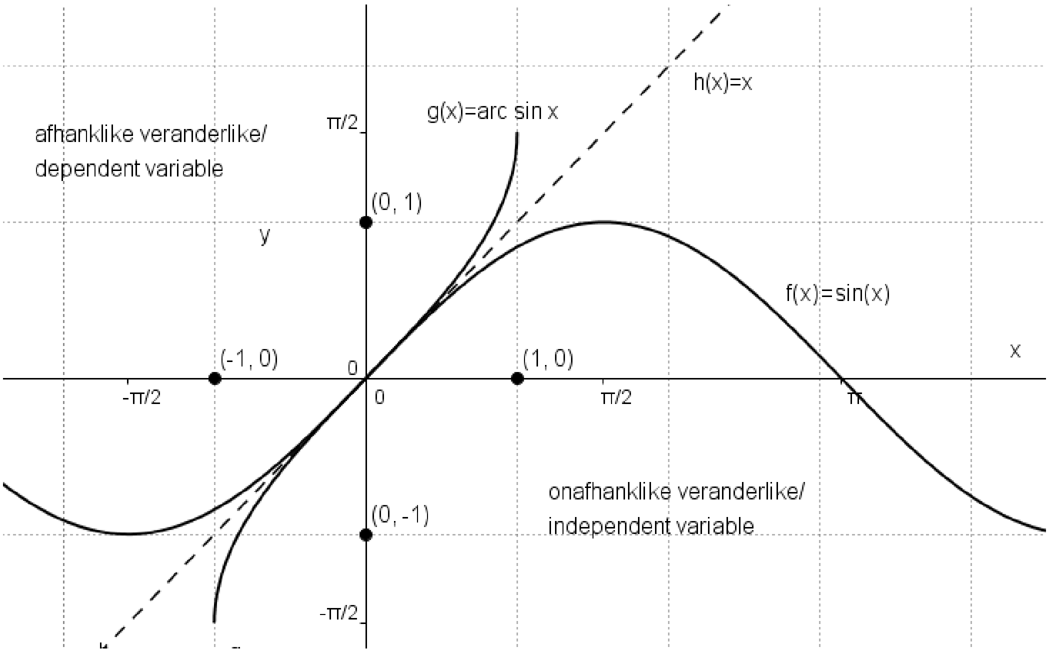
Calculation of inverse functions

In essence the calculation of the inverse of a function boils down to...

- that we switch the role of the dependent and independent variables in the defining equation and
- that we manipulate the resulting equation to make the vertical axis-variable (dependent variable) the subject of the equation.

There is, however, a complication involved with the calculation of inverse functions, and that is the fact that only one-to-one functions possess inverse functions. The inverse of a not one-to-one function, is a relation.

Consider, for example, the sine function f and its inverse f^{-1} which is indicated in the graph below as g.



Remark

Note that we conducted a similar argument in Study Section 2.1 where we introduced the notion of a logarithmic function as inverse of an exponential function.

There are, however, certain interesting aspects regarding the sine function (and the cosine function, too) which necessitate us to apply our deeper knowledge of functions when we calculate the inverse of certain functions.

The implication of one-to-one-ness when we determine the inverse of a function

It is clear that only the part of the sine curve between the points where it runs horizontally (in its turning points) possess a mirror image (reflection) in the line $y = x$.

Recall that a function $y = f^{-1}(x)$ connects by definition one and only one y-value with each x-value. Geometrically this means that a vertical line which moves over f^{-1}, may only intersect the curve once at every x-value.

But the inverse function f^{-1} was obtained from f by switching the roles of the dependent variable and the independent variable. Effectively we switched the axes. That implies that a horizontal line which moves over f, may intersect the curve only once at each y-value. Only that part of the range of f were it passes the horizontal line test, will form the domain of f^{-1}.

(Keep in mind that f^{-1} does NOT mean $\frac{1}{f}$ in this context.)

Exercise 3.3

Determine the inverse of the following functions and sketch f as well as f^{-1}. Also write down the domain and range

1. $f(x) = 2x - 1$
2. $g(x) = \sqrt{x - 1}$

3.4 Operations with functions

Combinations of functions (operations)

Since function values are numbers, functions behave like numbers with which we may perform operations.

The obvious complication, of course, is the domain of the result of the operation – **in general the domain of the result will be the intersection of the domains of the separate constituent functions** – in the case of division (quotients) **the values of the independent variable for which the denominator function is zero**, will also be excluded.

Take note of the following notation.

- Sum of functions: $(f+g)(x)=f(x)+g(x)$
- Difference of functions: $(f-g)(x)=f(x)-g(x)$
- Product of functions: $(fg)(x)=f(x)g(x)$
- Quotient of functions: $\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}$, $g(x)\neq 0$.

Exercise 3.4

1. If $f(x)=2x^2$ and $g(x)=3x+5$, determine the following.

   1.1. $(f+g)(x)$

   1.2. $(g+f)(x)$

   1.3. $(f-g)(x)$

   1.4. $(g-f)(x)$

   1.5. $(fg)(x)$

   1.6. $(gf)(x)$

   1.7. $\left(\frac{f}{g}\right)(x)$

   1.8. $\left(\frac{g}{f}\right)(x)$

2. Which of the operations above are commutative?

3. If $f(x)=2x^2$, $a=3$ and $x=2$ determine the following.

   3.1. $a[f(x)]$

   3.2. $f(ax)$

   3.3. What conclusion can you draw about 3.1 and 3.2?

4. If $g(x)=2x^2$, $a=9$ and $x=16$, determine the following.

   4.1. $g(a+x)$

   4.2. $g(a)+g(x)$

   4.3. What conclusion can you draw about 4.1 and 4.2?

5. If $p(x)=\sin x$ and $a=4$, sketch the following three curves on the same set of axes using a graphing calculator or graphing software.

$$y=p(x),\qquad y=ap(x),\qquad y=p(ax)$$

3.5 Composite functions

A composite function may be considered as a function within another function. Some writers refer to this phenomenon as "nested functions". Whenever we evaluate such a function, we proceed from the inside out.

Example 1

The volume of a spherical balloon is given by the formula $V(r) = \frac{4}{3}\pi r^3$. The radius changes with time according to the formula $r(t) = -t^2 + 6t$ with $0 \leq t \leq 3$ where the radius is measured in cm and the time in seconds. We wish to calculate the volume at the instant when $t = \frac{3}{2}$. Now we may write the volume as

$$(V \circ r)(t) = V(r(t)) = V(-t^2 + 6t) = \underbrace{\frac{4}{3}\pi \left(\underbrace{-t^2 + 6t}_{r(t)} \right)^3}_{V(r)}$$

We could attempt to simplify the expression on the right by multiplying out the parentheses until we obtain the volume as a polynomial function $V(t)$. Then, we could simply substitute $t = \frac{3}{2}$ in order to obtain $V\left(\frac{3}{2}\right)$. This approach, however, would generate a lot of tedious calculation.

The theory of composite functions permits us to rather proceed as follows. Compute $r\left(\frac{3}{2}\right)$ and substitute the answer into $V(r) = \frac{4}{3}\pi r^3$; then we also obtain the value of V when $t = \frac{3}{2}$.

Complete: $r\left(\frac{3}{2}\right) = \cdots$, $V(\dots) = \frac{4}{3}\pi(\dots)^3 = \cdots$.

What is the domain of r?

What is the range of r?

What is the domain of V?

What is the range of V?

Example 2

Given $S(r) = 4\pi r^2$ and $r(t) = t^2 + 5t + 5$ with $0 \leq t \leq 10$.

We wish to calculate the surface S when $t = 6$.

Complete: $(S \circ r)(t) = \dots\dots\dots\dots = \underbrace{4\pi \left(\underbrace{\dots\dots\dots\dots}_{r(t)} \right)^2}_{S(r)}$

$r(6) = \cdots$, $S(\dots) = 4\pi(\dots)^2 = \cdots$.

What is the domain of r?

What is the range of r?

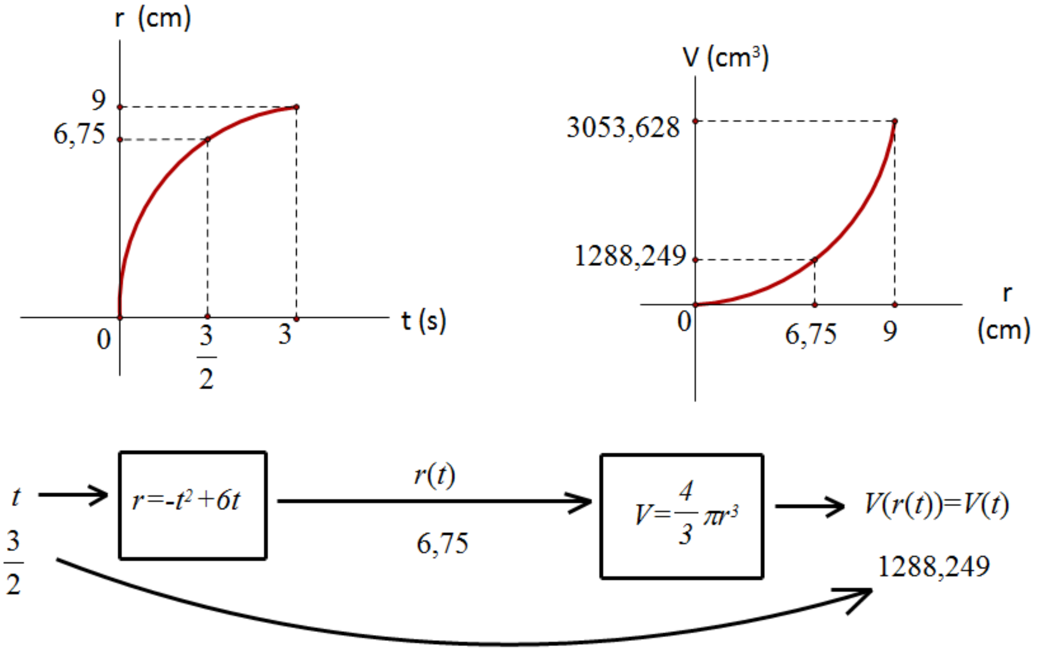
What is the domain of S?

What is the range of S?

Definition of a composite function

If f and g are functions of x, then the composite function $(f \circ g)(x)$ is the nested function $f\big(g(x)\big)$ for all x in the domain of g in order that $g(x)$ forms the domain of f.

Schematically we can represent the example about the balloon (Example 1 above) as follows.



In the application of the **chain rule for derivatives** which you will study in detail later this year, you must be able to identify the inner and outer constituents of a composite function.

Example 1

If $f(x) = x^2$ and $g(x) = \sqrt{x+2}$ then $(f \circ g)(x) = f\big(g(x)\big) = \big(\sqrt{x+2}\big)^2 = x + 2$ and $(g \circ f)(x) = g\big(f(x)\big) = \sqrt{(x^2) + 2} = \sqrt{x^2 + 2}$.

Example 2

Resolve $f(x) = \cos(x^2 - x + 1)$ into the functions $u(x)$ and $v(x)$.

Take $u(x) = \cos x$ and $v(x) = x^2 - x + 1$. Then $f(x) = u\big(v(x)\big) = (u \circ v)(x)$.

Exercise 3.5

1. If $v(t) = \sqrt[3]{t}$ and $u(t) = \sin t$, determine the following.

    1.1. $(v \circ u)(t)$

    1.2. $(u \circ v)(t)$

2. If $v(t) = \frac{2}{1+t}$ and $u(t) = \frac{3}{1-t}$, determine the following.

    2.1. $(v \circ u)(t)$

    2.2. $(u \circ v)(t)$

4 Radian measure and trigonometry

Learning aims for this study unit

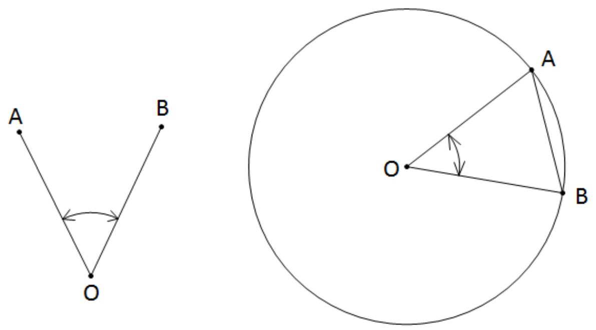
Upon completion of this study unit the student must be able to do the following:

1. Define radian measure and convert angles form degrees to radians and vice versa.
2. Calculate arc length.
3. Calculate the area of a circle sector.
4. Define all six trigonometric ratios and calculate their function values in all four quadrants of the flat plane.
5. Apply the sum and difference formulas.
6. Apply the double angle formulas.
7. Prove trigonometric identities.

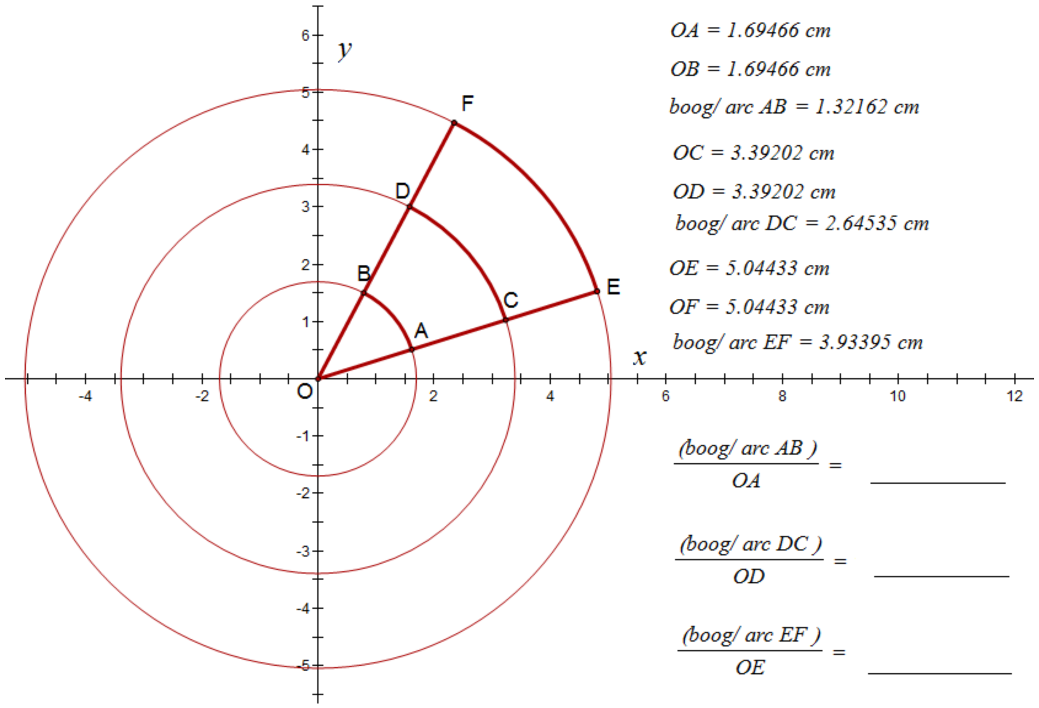
4.1 Radian measure

The subdivision of a circle into 360 equal parts which we call degrees, as well as the subdivision of a degree into 60 equal parts which we call minutes, is an ancient tradition which were passed on from the Babylonian and Sumerian scholars of earlier than 540 B.C.

Use the following sketches and attempt to explain what we mean by the concept "angle".



For modern Mathematics and Science we require a formal definition for the concept **angle**. Consider the sketch below where accurate measurements are shown. Calculate the ratio arclength to radius for all three sectors.



Note that the ratio arc length to radius remains constant in all three cases, irrespective of the area between the lines and/or the arcs defining the sectors. That is why it is useful to henceforth simply define the angle between any two radii from now on as

$$\theta = \frac{\text{arc length}}{\text{radius}} = \frac{a}{r}.$$

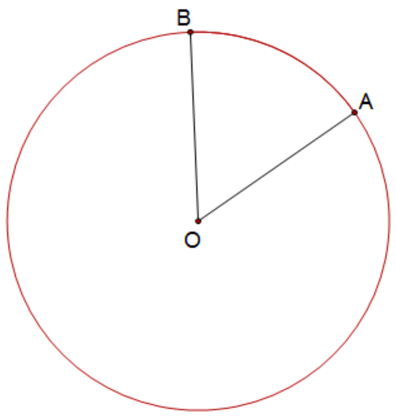
Note that the angle θ does not possess any units. Why not? We refer to angle size calculated in this way as **radians**.

How big is one radian and how many radians fit into a full circle, which we traditionally consider as the arc of one revolution (360°)?

Consider the following sketch and complete the information on it.

boog/ arc AB = 3.206 cm

radius = 3.206 cm



$$\frac{\text{boog/ arc AB}}{\text{radius}} = \underline{\hspace{5cm}}$$

You can see that an angle of 1 radian is a rather large angle. Estimate the size of the angle in degrees. Let us now develop a method to convert radians to degrees.

Consider a circle with radius r and circle arc a where the arc is in this case the entire circumference of the circle. In this case the angle which is subtended by the arc (the circumference) is one revolution. It is 360° in terms of traditional units.

Complete the reasoning.

$$\text{One rev}\left(^\circ\right) = 1\,\text{Rev}\left(\text{radians}\right)$$

$$\therefore 360^\circ = \frac{\textit{arc length}\ (\textit{full circle})}{\textit{radius}} \qquad (\text{by def.})$$

$$\therefore 360^\circ = \frac{\dots\dots\dots\dots\dots\dots}{r} \qquad (\text{circumf=?})$$

$$\therefore 360^\circ = \dots\dots\dots\dots\dots \quad \text{radians}$$

$$\therefore 1^\circ = \dots\dots\dots\dots\dots \quad \text{radians}$$

$$1\,\text{radian} = \dots\dots\dots\dots{}^\circ$$

It is now easy to see why we may consider an angle of π radians as an angle of 180°.

Example

1. Convert 240° to radians.
2. Convert 315° to radians.
3. Convert $\frac{3}{4}\pi$ radians to degrees.
4. Convert $\frac{11}{6}\pi$ radians to degrees.

Solution

1. $$\begin{aligned}240^\circ &= 180^\circ + 60^\circ \\ &= \pi\ rad + \frac{1}{3}\pi\ rad \\ &= \frac{3}{3}\pi\ rad + \frac{1}{3}\pi\ rad \\ &= \frac{4}{3}\pi\ rad\end{aligned}$$

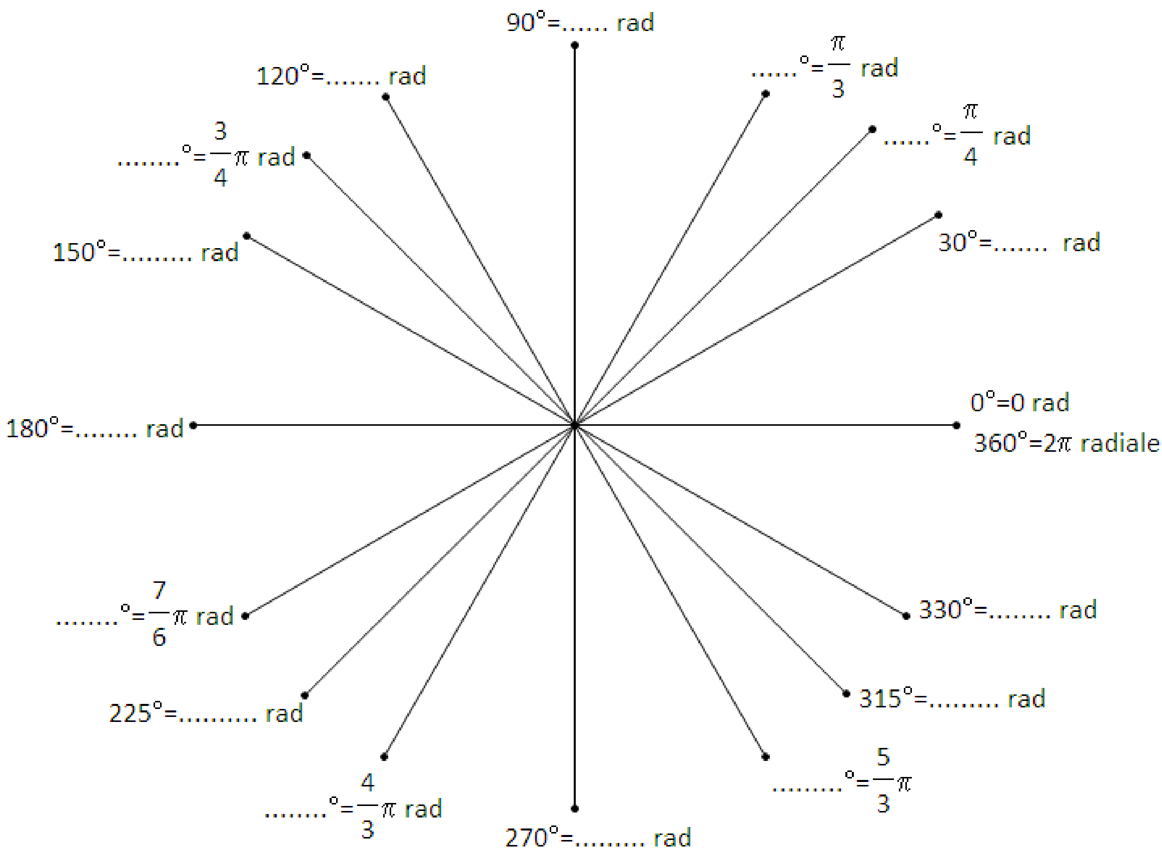
2. $$\begin{aligned}315^\circ &= 360^\circ - 45^\circ \\ &= 2\pi\ rad - \frac{1}{4}\pi\ rad \\ &= \frac{8}{4}\pi\ rad - \frac{1}{4}\pi\ rad \\ &= \frac{7}{4}\pi\ rad\end{aligned}$$

3. $$\begin{aligned}\frac{3}{4}\pi\ rad &= \frac{3}{4}\times 180^\circ \\ &= 135^\circ\end{aligned}$$

4. $$\begin{aligned}\frac{11}{6}\pi\ rad &= \frac{11}{6}\times 180^\circ \\ &= 330^\circ\end{aligned}$$

Exercise 4.1

Complete the omitted information in the sketch below.



4.2 Calculation of arc length

Solve the following problems using the definition of angle size in radians. Where angle sizes are given in degrees, first convert them to radians before doing the calculations. If the answer is an angle size, convert it from radians to degrees.

Exercise 4.2

1. A toy train travels around a circular track with an area of $25\ 446.900$ cm². If it covers 2.5 m between two points on the track, what is the angle in degrees through which the train travelled?
2. Suppose a circle sector has a radius of x and a central angle of 2 radians. Calculate the length of the arc of the sector.
3. Determine the area of a circle if an arc of 200 mm is subtended by an angle of $171.887339°$.

4.3 Calculation of the area of a circle sector

Consider the following table and see if you can discover the pattern.

Figure	Area formula	Angle in rad
Full circle	$A = \pi r^2$	2π
Semi-circle	$A =$ ....πr^2	π
Quarter circle	$A =$ ....πr^2	....
Eighth of circle	$A = \frac{1}{8}\pi r^2$	$\frac{\pi}{4}$
n^{th} of a circle	$A =$ ....πr^2	....

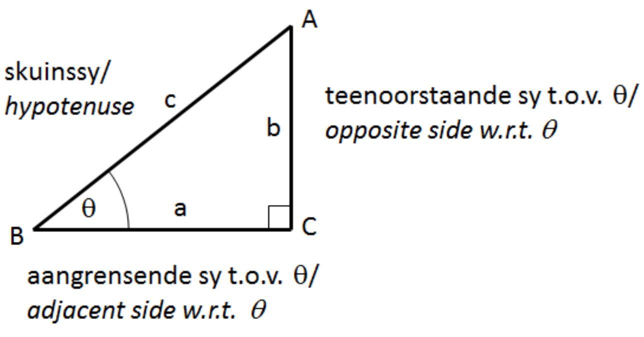
Complete: If a circle sector makes an angle of θ radians at the centre, the formula for the area of the sector is $A_{\text{circle sector}} = \cdots$.

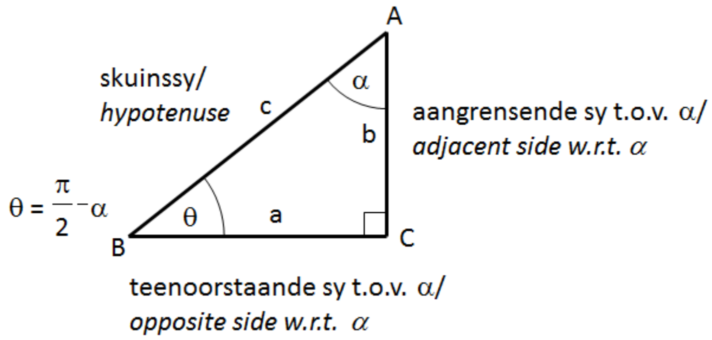
Exercise 4.3

1. The jet of a water sprayer sweeps through an angle of $50°$ and the jet can reach 5 m far. Calculate the area which the sprayer can irrigate.
2. Suppose the angle θ through which the jet sweeps, doubles but the radius halves. By which percentage would the area which can be irrigated, then change? Is this an increase or decrease?
3. Suppose the angle θ through which the jet sweeps, halves but the radius doubles. By which percentage would the area which can be irrigated, then change? Is this an increase or decrease?
4. Determine the radius of a circle whose area of an eighth of the circle is 19.2422555 m².

4.4 The six trigonometric ratios and their function values in all four quadrants of the flat plane

Consider the right triangle.





Since there exist six different ways in which to write ratios for the opposite side, adjacent side and hypotenuse of a right triangle, not only the following ratios hold true

$$\sin\theta = \frac{b}{c} \qquad \text{en/and} \qquad \sin\alpha = \frac{\ldots\ldots\ldots}{\ldots\ldots\ldots}$$

$$\cos\theta = \frac{\ldots\ldots\ldots}{\ldots\ldots\ldots} \qquad \text{en/and} \qquad \cos\alpha = \frac{\ldots\ldots\ldots}{c}$$

$$\tan\theta = \frac{\ldots\ldots\ldots}{\ldots\ldots\ldots} \qquad \text{en/and} \qquad \tan\alpha = \frac{\ldots\ldots\ldots}{\ldots\ldots\ldots}$$

but also the following ratios

$$\text{cosec}\,\theta = \frac{1}{\sin\theta} = \frac{c}{b} \qquad \text{en/and} \qquad \text{cosec}\,\alpha = \frac{1}{\sin\alpha} = \frac{\ldots\ldots\ldots}{\ldots\ldots\ldots}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{\ldots\ldots\ldots}{\ldots\ldots\ldots} \qquad \text{en/and} \qquad \sec\alpha = \frac{1}{\cos\alpha} = \frac{\ldots\ldots\ldots}{\ldots\ldots\ldots}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{\ldots\ldots\ldots}{\ldots\ldots\ldots} \qquad \text{en/and} \qquad \cot\alpha = \frac{1}{\tan\alpha} = \frac{\ldots\ldots\ldots}{\ldots\ldots\ldots}$$

The ratios in the second group are known as **reciprocal ratios of the sine, cosine and tangent ratios**. **(NOT INVERSE RATIOS)**

IT can easily be shown that $\tan\theta = \frac{\sin\theta}{\cos\theta}$ by using the first sketch.

$$\text{LHS} = \tan\theta = \frac{b}{a}$$

$$\text{RHS} = \frac{\sin\theta}{\cos\theta} = \frac{b/c}{a/c} = \frac{b}{c} \times \frac{c}{a} = \frac{b}{a}$$

Therefore $\text{LK} = \text{RK}$ and $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

Now make use the second sketch and prove on your own that $\cot\theta = \frac{\cos\theta}{\sin\theta}$.

The second sketch also provides us with a way to obtain **the co-ratios of the six trigonometric ratios**. Complete the following.

$\sin\alpha = \frac{\ldots\ldots}{\ldots\ldots}$ and $\cos\theta = \frac{\ldots\ldots}{\ldots\ldots}$ therefore ……………. = …………….. . But, since $\theta = \frac{\pi}{2} - \alpha$, we can write

$$\sin \ldots = \cos(\ldots\ldots\ldots).$$

$\cos\alpha = \frac{\ldots\ldots}{\ldots\ldots}$ and $\sin\theta = \frac{\ldots\ldots}{\ldots\ldots}$ therefore ……………. = …………….. . But, since $\theta = \frac{\pi}{2} - \alpha$, we can write

$$\cos \ldots = \sin(\ldots\ldots\ldots).$$

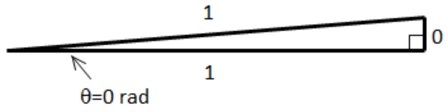
$\tan\alpha = \frac{\ldots\ldots}{\ldots\ldots}$ and $\cot\theta = \frac{\ldots\ldots}{\ldots\ldots}$ therefore ……………. = …………….. . But, since $\theta = \frac{\pi}{2} - \alpha$, we can write

$$\tan \ldots = \cot(\ldots\ldots\ldots).$$

$\sec\alpha = \frac{\ldots\ldots}{\ldots\ldots}$ and $\text{cosec}\,\theta = \frac{\ldots\ldots}{\ldots\ldots}$ therefore ……………. = …………….. . But, since $\theta = \frac{\pi}{2} - \alpha$, we can write

$$\sec \ldots = \text{cosec}(\ldots\ldots\ldots).$$

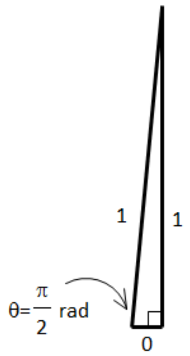
It is also known to you that the function values for **certain familiar angles**, namely. $0°$, $30°$, $45°$, $60°$ and $90°$ may very easily be obtained from the following sketches and may also be easily memorized; We adjusted the sketches below for radian measure. Complete the omitted information.



$\sin 0 =$ $\quad$ $\text{cosec}\ 0 =$

$\cos 0 =$ $\quad$ $\sec 0 =$

$\tan 0 =$ $\quad$ $\cot 0 =$

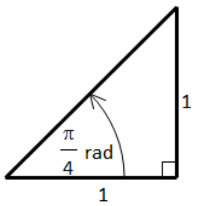


$\sin \frac{\pi}{2} =$ $\quad$ $\text{cosec}\ \frac{\pi}{2} =$

$\cos \frac{\pi}{2} =$ $\quad$ $\sec \frac{\pi}{2} =$

$\tan \frac{\pi}{2} =$ $\quad$ $\cot \frac{\pi}{2} =$

Note that the triangles above do not really exist. We use imagination to determine what happens to the triangle if the side and the hypotenuse of the triangle coincide.



$\sin \frac{\pi}{4} =$ $\quad$ $\text{cosec}\ \frac{\pi}{4} =$

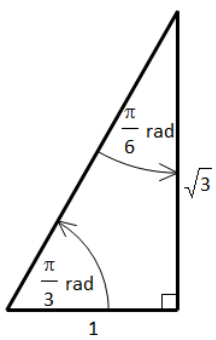
$\cos \frac{\pi}{4} =$ $\quad$ $\sec \frac{\pi}{4} =$

$\tan \frac{\pi}{4} =$ $\quad$ $\cot \frac{\pi}{4} =$

$\sin \frac{\pi}{3} =$ $\quad$ $\text{cosec}\ \frac{\pi}{3} =$

$\cos \frac{\pi}{3} =$ $\quad$ $\sec \frac{\pi}{3} =$

$\tan \frac{\pi}{3} =$ $\quad$ $\cot \frac{\pi}{3} =$

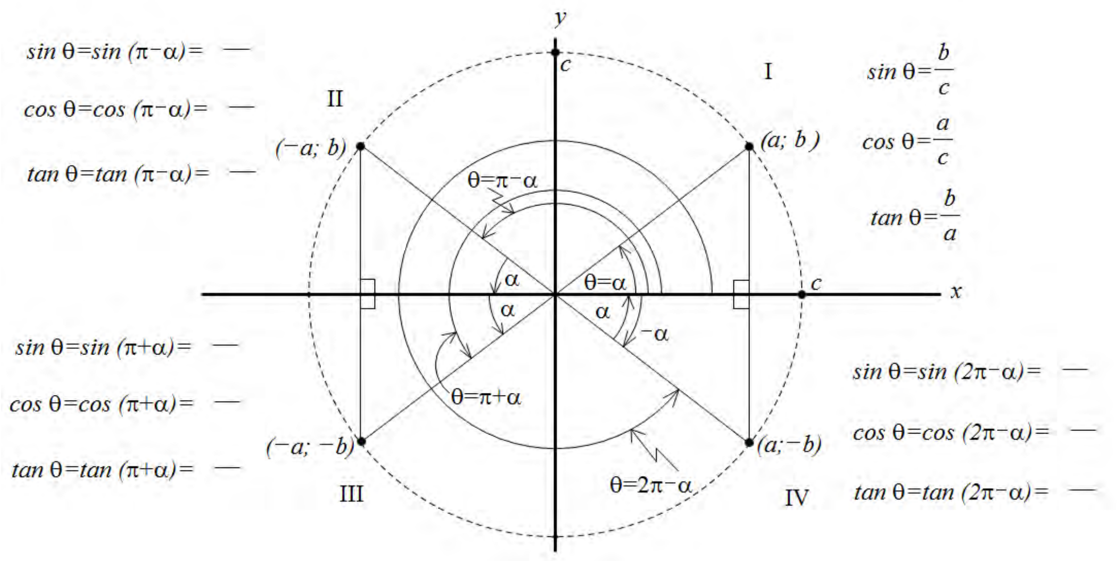


$\sin \frac{\pi}{6} =$ $\quad$ $\text{cosec}\ \frac{\pi}{6} =$

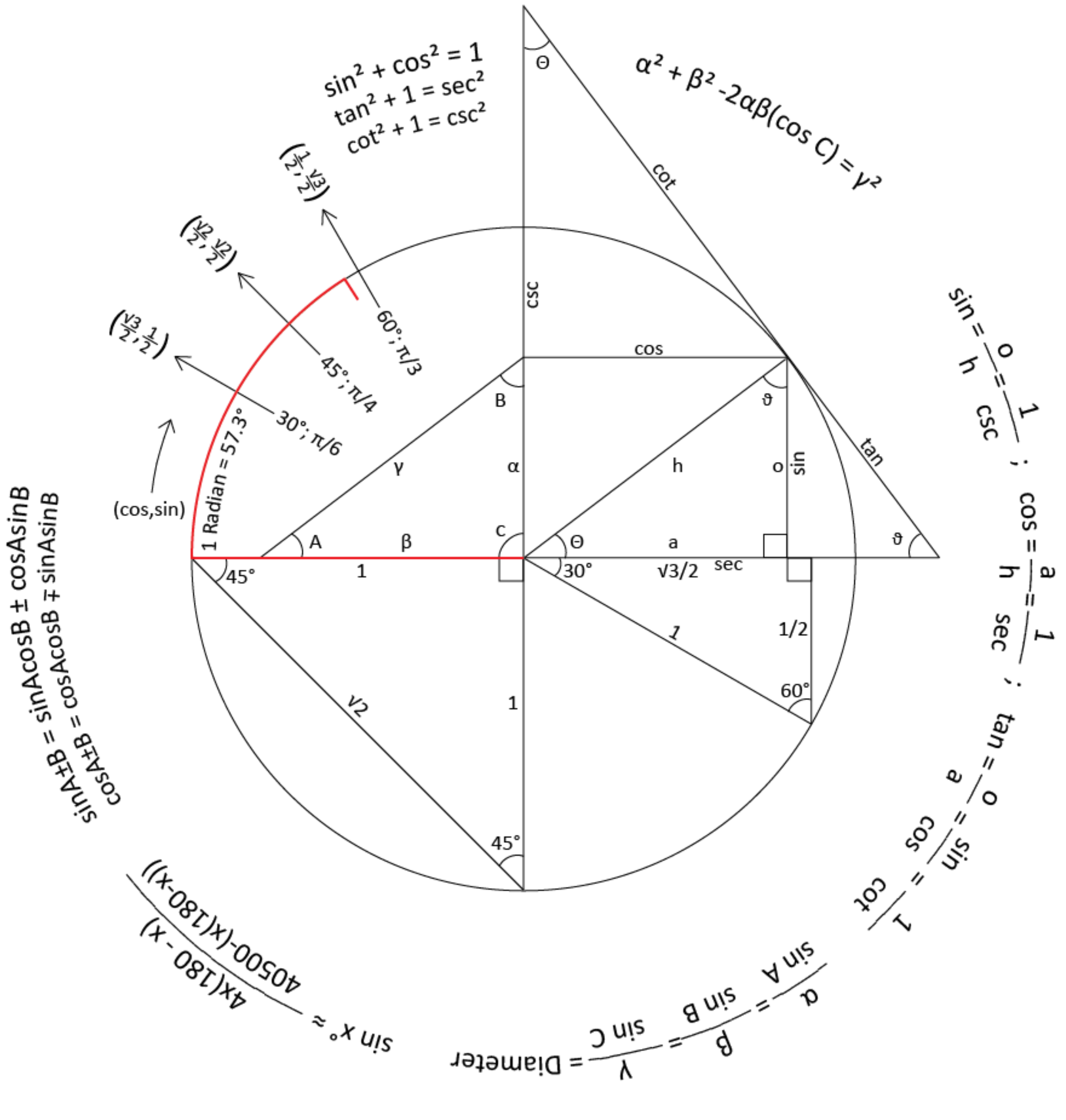
$\cos \frac{\pi}{6} =$ $\quad$ $\sec \frac{\pi}{6} =$

$\tan \frac{\pi}{6} =$ $\quad$ $\cot \frac{\pi}{6} =$

Because it often happens that an angle is larger than $\frac{\pi}{2}$ radians, **we must be able to do trigonometry with angles in four quadrants of the flat plane**. Complete the sketch below.



Below you find another interesting way of summarizing most of our trigonometric knowledge.



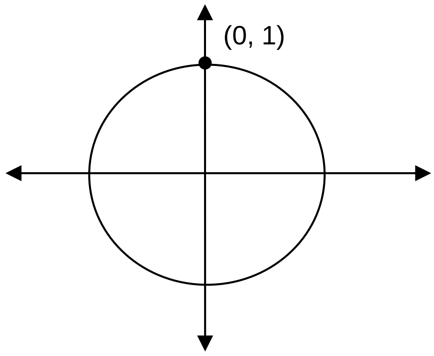
More on special angles

Use the definitions in terms of a unit circle and find the values of sine, cosine and tangent for the following **special angles** using the sketches supplied below.

(a) $\cos \frac{\pi}{2}$ = ………………………………

(b) $\sin \frac{\pi}{2}$ = ………………………………

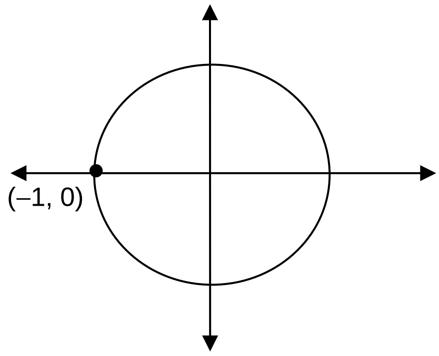
(c) $\tan \frac{\pi}{2}$ = ………………………………



(d) $\cos \pi$ = …………………………………

(e) $\sin \pi$ = ……………………………………..

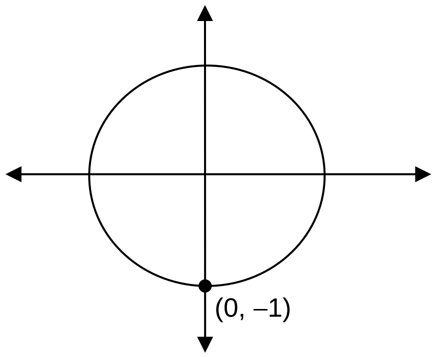
(f) $\tan \pi$ = ……………………………..……..



(g) $\cos \frac{3\pi}{2}$ = ………..………..………

(h) $\sin \frac{3\pi}{2}$ = ……………………..……….

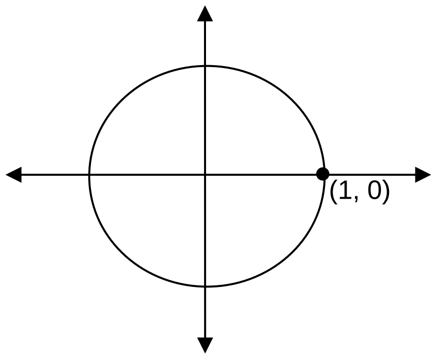
(i) $\tan \frac{3\pi}{2}$ = …………………………………



(j) $\cos 2\pi$ = ……………………..………….

(k) $\sin 2\pi$ = …………………………………….

(l) $\tan 2\pi$ = ……………………………………



Exercise 4.4

1. Calculate without a calculator and without converting the angle sizes to degrees, the values of the following expressions.

   1.1. $\sqrt{\tan^2\left(\frac{4}{3}\pi\right)+\sec^2\left(\frac{3}{4}\pi\right)\times 4\cos^2\left(\frac{11}{6}\pi\right)}$

   1.2. $\left(\sin\left(\frac{2}{3}\pi\right)+\cos\left(\frac{5}{3}\pi\right)\right)^2+\text{cosec}^2\left(\frac{\pi}{6}\right)$

   1.3. $\sqrt{\cos\left(\frac{\pi}{2}\right)+\sin\left(\frac{5}{2}\pi\right)+3\cos(2\pi)}$

   1.4. $3\cos^2(2\pi)+\frac{1}{3}\tan^2\left(\frac{4}{3}\pi\right)+2\sin\left(\frac{5}{6}\pi\right)+2\cos^5\left(\frac{3}{2}\pi\right)$

   1.5. $\dfrac{\sin^2\left(\frac{5}{4}\pi\right)\tan\left(\frac{4}{3}\pi\right)\sec^2\left(\frac{11}{6}\pi\right)}{\tan\left(\frac{2}{3}\pi\right)}$

   1.6. $\sin^2\left(\frac{4}{3}\pi\right)+\sin^2\left(\frac{7}{6}\pi\right)$

   1.7. $\sec^2\left(\frac{\pi}{3}\right)-\tan^2\left(\frac{\pi}{3}\right)$

   1.8. $\text{cosec}^2\left(\frac{2}{3}\pi\right)-\cot^2\left(\frac{2}{3}\pi\right)$

2. Use the sketches at the beginning of this study section and prove the following.

   2.1. $\text{cosec}^2\theta=\cot^2\theta+1$

   2.2. $\sec^2\theta=\tan^2\theta+1$

   2.3. $\tan^4\theta+\tan^2\theta=\sec^4\theta-\sec^2\theta$

3. Simplify to simplest form.

   3.1. $\dfrac{\sin(\pi-\theta)\cot(\pi+\theta)\sec(2\pi-\theta)}{\cos^2(\pi+\theta)+\cos^2\left(\frac{\pi}{2}-\theta\right)}$

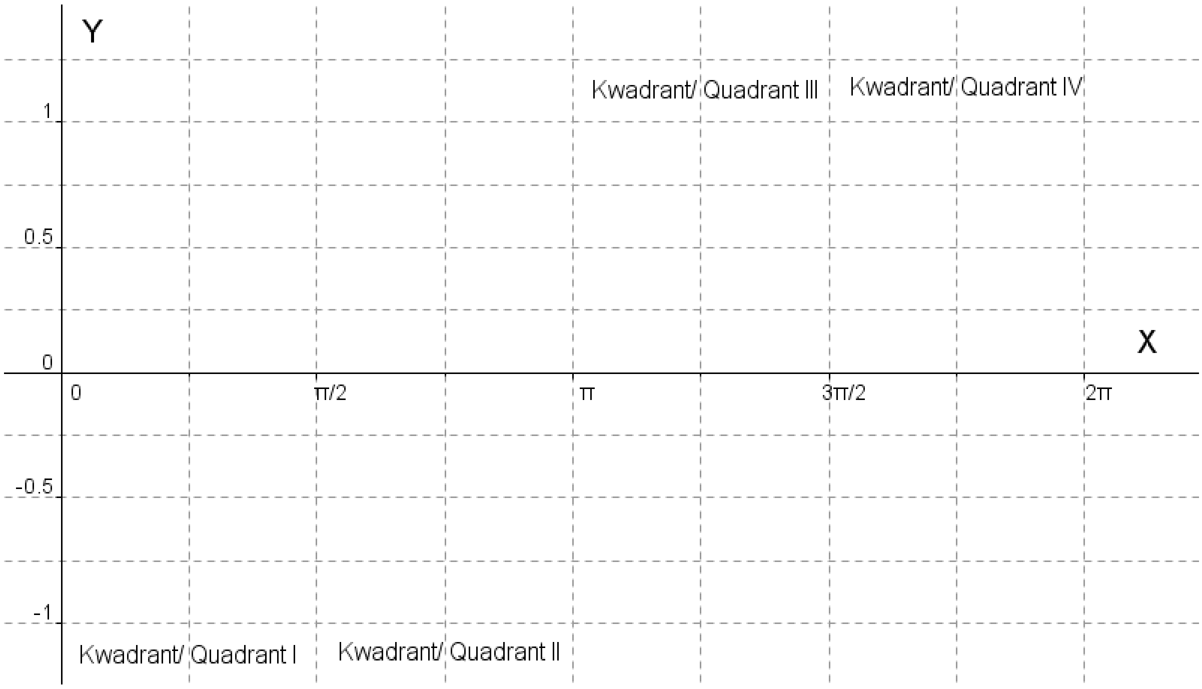
   3.2. $\dfrac{\tan(\pi-\theta)\sqrt{1-\sin^2\theta}}{\cos\left(\frac{\pi}{2}-\theta\right)}$

   3.3. $\dfrac{\sin(\pi+A)\cos(2\pi-A)}{\cos\left(\frac{\pi}{2}-A\right)\cos(2\pi+A)}$

4. Complete the following three representations of the basic trigonometric graphs.

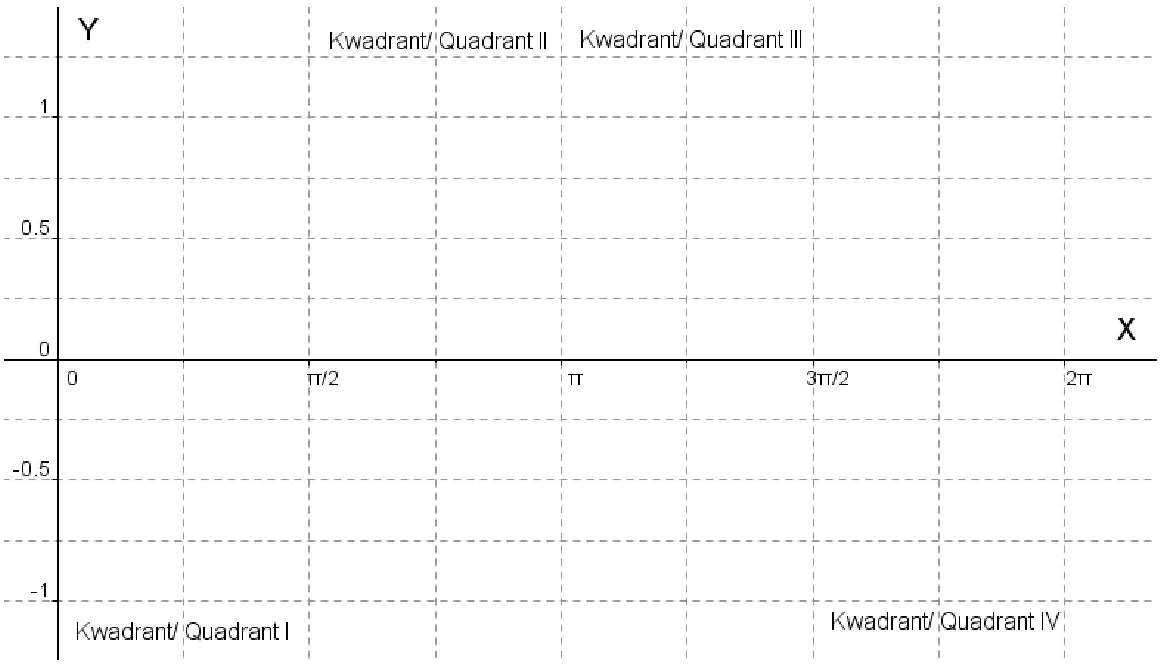
4.1. $y = \sin x$ for $x \in [0, 2\pi]$

x	0	$\frac{1}{4}\pi$	$\frac{1}{2}\pi$	$\frac{3}{4}\pi$	π	$\frac{5}{4}\pi$	$\frac{3}{2}\pi$	$\frac{7}{4}\pi$	2π
y									



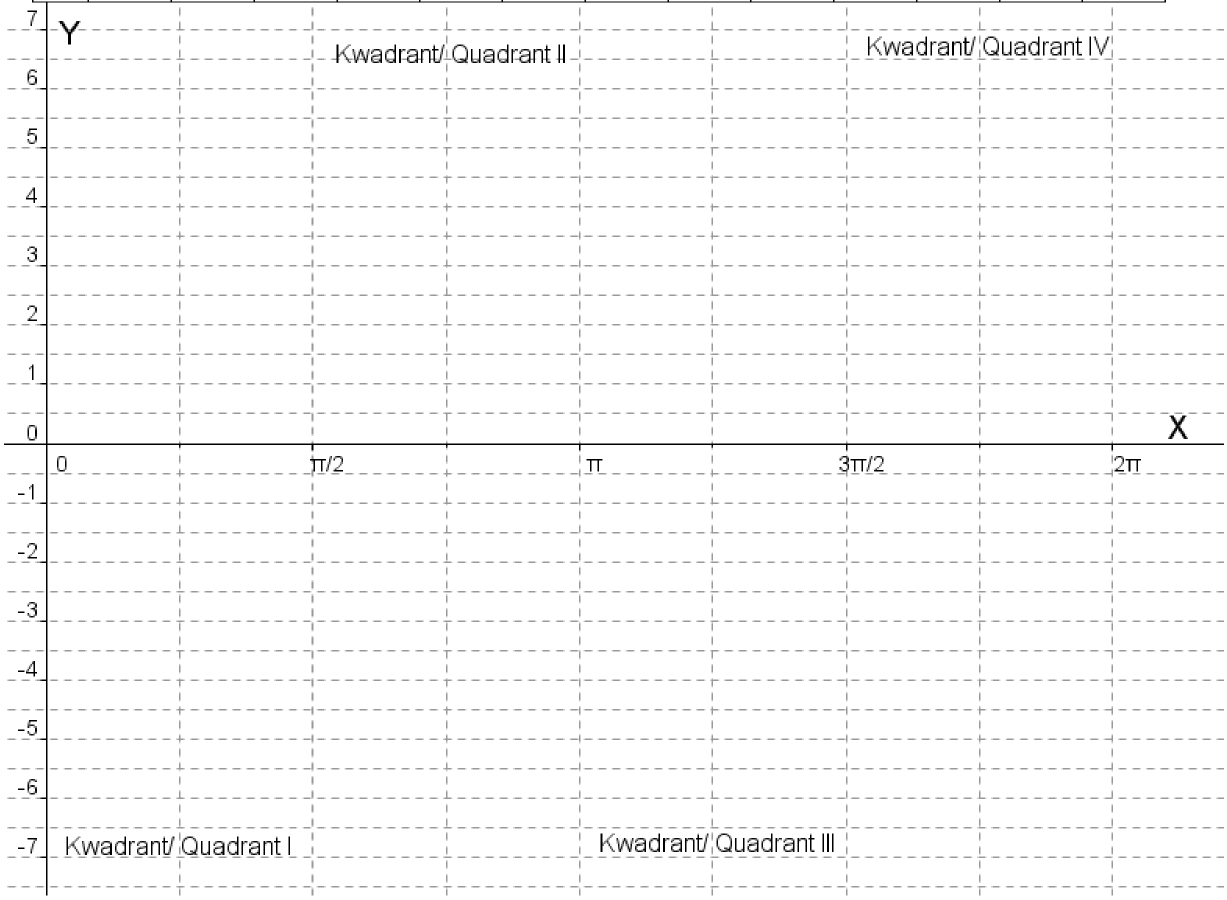
4.2. $y = \cos x$ for $x \in [0, 2\pi]$

x	0	$\frac{1}{4}\pi$	$\frac{1}{2}\pi$	$\frac{3}{4}\pi$	π	$\frac{5}{4}\pi$	$\frac{3}{2}\pi$	$\frac{7}{4}\pi$	2π
y									



4.3. $y = \tan x$ for $x \in [0,2\pi]$

x	0	$\frac{\pi}{4}$	$\frac{45\pi}{100}$	$\frac{\pi}{2}$	$\frac{55\pi}{100}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{145\pi}{100}$	$\frac{3\pi}{2}$	$\frac{155\pi}{100}$	$\frac{7\pi}{4}$	2π
y													



5. Solve the following trigonometric equations for values of θ such that $\theta \in [0,2\pi]$.

5.1. $\sin\theta = -\frac{1}{2}$

5.2. $2\cos\theta = \sqrt{3}$

5.3. $\sqrt{3}\tan\theta + 1 = 0$

5.4. $\sec 2\theta + 2 = 0$

5.5. $\sqrt{3}\csc 2\theta + 2 = 0$

6. Solve the following trigonometric equations for all radial values of θ, in other words, find the general solutions.

6.1. $\cos\left(x + \frac{\pi}{3}\right) = \sin\left(3x - \frac{\pi}{3}\right)$

6.2. $\sin(\cos x) = 0$

4.5 The sum and difference formulae

From the definitions of the trigonometric functions the following important identities may be derived; in this course we do not have time to discuss their derivations, but you may research that on your own.

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

Exercise 4.5

1. Given that $\sin A = \frac{5}{13}$, $A \in \left[0, \frac{\pi}{2}\right]$ and $\cos B = -\frac{3}{5}$, $B \in \left[\pi, \frac{3\pi}{2}\right]$. Calculate the values of the expressions without using a calculator.

   1.1. $\sin(A - B)$

   1.2. $\cos(A + B)$

   1.3. $\tan(B - A)$

2. Without using a calculator, calculate the value of $\cos\frac{\pi}{2}$ by using $\cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$.
3. Prove that $\tan(\pi + \theta) = \tan\theta$.
4. Without using a calculator, calculate the value $\cos 23° \cos 67° - \sin 23° \sin 67°$.
5. Without using a calculator, calculate the value $\frac{\tan 18° + \tan 27°}{1 - \tan 18° \tan 27°}$.

4.6 The double angle formulae

From the sum and difference formulae the following important identities may be derived.

$$\sin 2A = 2\sin A \cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A$$
$$= 2\cos^2 A - 1$$
$$= 1 - 2\sin^2 A$$
$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Exercise 4.6

1. Prove that $\sin 2A = 2 \sin A \cos A$.
2. Prove that $\cos 2A = 1 - 2\sin^2 A$.
3. Use the expression in Question 2 and derive an expression for $\sin^2 A$ in terms of $\cos 2A$.
4. Prove that $\cos 2A = 2\cos^2 A - 1$.
5. Use the expression in Question 4 and derive an expression for $\cos^2 A$ in terms of $\cos 2A$.
6. Prove the following identities.

   6.1. $\frac{\sin\phi}{1+\cos\phi} = \tan\phi$

   6.2. $\sin(x + y)\sin(x - y) = \sin^2 x - \sin^2 y$

   6.3. $\frac{\sec x - \cos x}{\sin x + \tan x} = \operatorname{cosec} x - \cot x$

7. Prove that $\cos\frac{25\pi}{6} = \cos\frac{\pi}{6}$.
8. Prove that $\cos\frac{28\pi}{3} = \cos\left(-\frac{2\pi}{3}\right)$.

4.7 Application of trigonometry and trigonometric functions

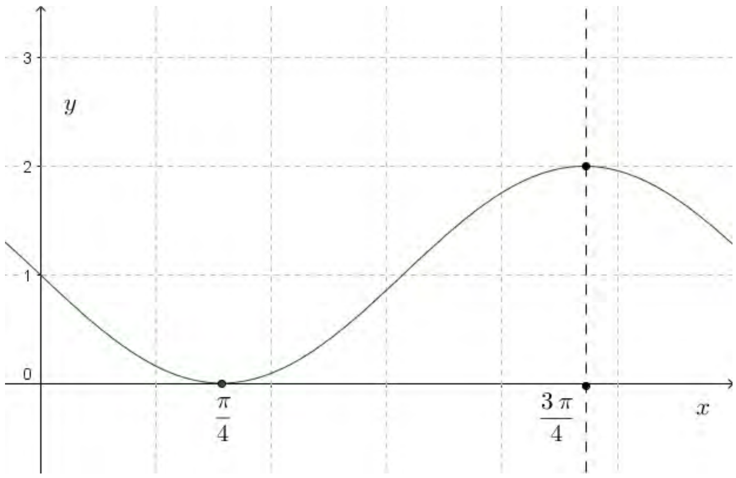
Exercise 4.7

1. A triangular parking lot has dimensions $\hat{B} = 29.5°$, $a = 254$ cm and $b = 195$ cm. Determine the area of this triangle if $\hat{C}$ is an acute angle.

2. A nautical mile is defined as the distance along an arc on the surface of the earth that subtends a central angle of 1 minute (1 min = 1/60 degrees). Find the value of a nautical mile if the radius of the earth is 3960 miles.

3. A shipping vessel leaves Durban harbour on a bearing of N 59º E and sails 35 nautical miles in the direction of Richards Bay. It then turns to a bearing of S 52º E and sails 22 nautical miles in this direction. What is the direct distance between the vessel and Durban at this point??

4. Consider the graph below.



4.1. Determine an equation for the curve in each of the forms.

4.1.1. $y = a\sin(bx + c) + d$

4.1.2. $y = a\sin[\omega(x + p)] + d$

4.1.3. $y = a\cos(bx + c) + d$

4.1.4. $y = a\cos[\omega(x + p)] + d$

4.2. Is the function even, odd function or neither?

4.3. Classify the shifted graphs as even, odd function or neither:

4.3.1. The graph is shifted downwards by 1 unit.

4.3.2. The graph is shifted upwards by 1 unit.

4.3.3. The graph is shifted to the left by $\frac{\pi}{4}$ units.

5. A plastic holder floating in the ocean is bobbing in simple harmonic motion. Its displacement above the ocean floor is modelled by $y = 0.2\cos(20\pi t) + 8$ with y in meters and t in minutes.

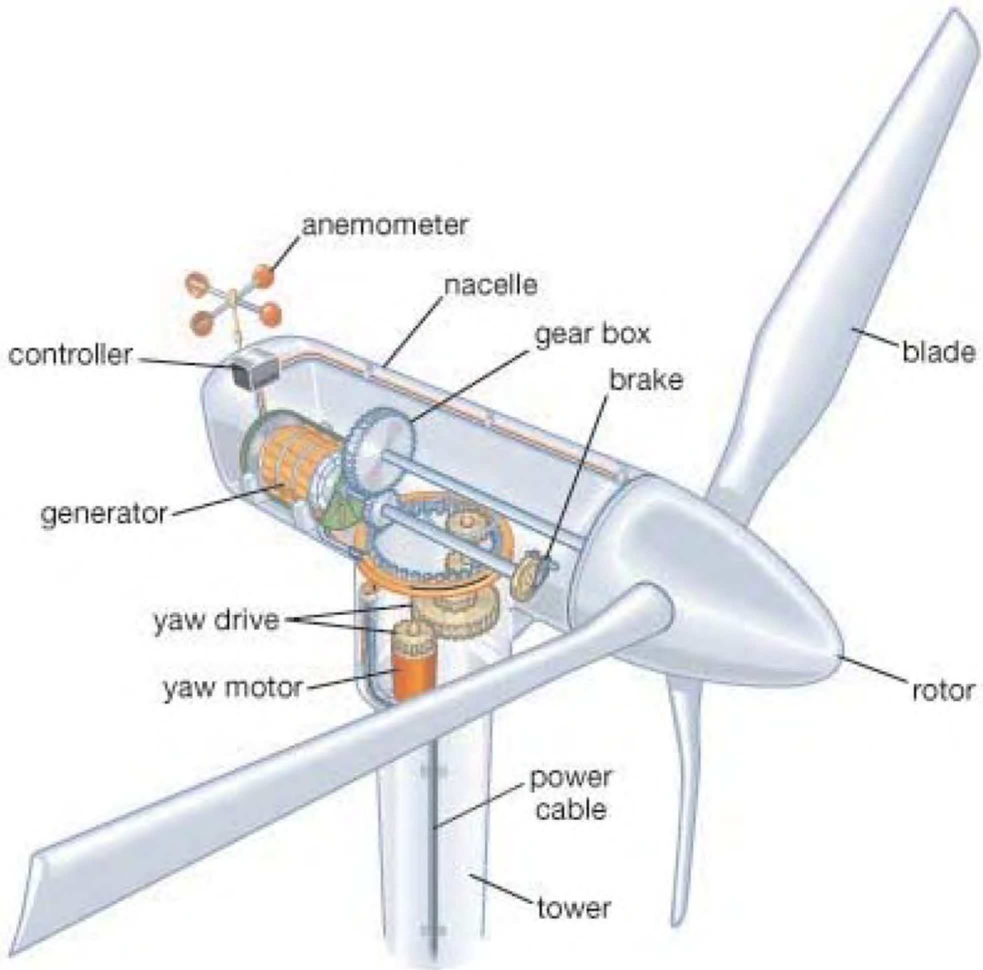
5.1. Determine the frequency of the movement.

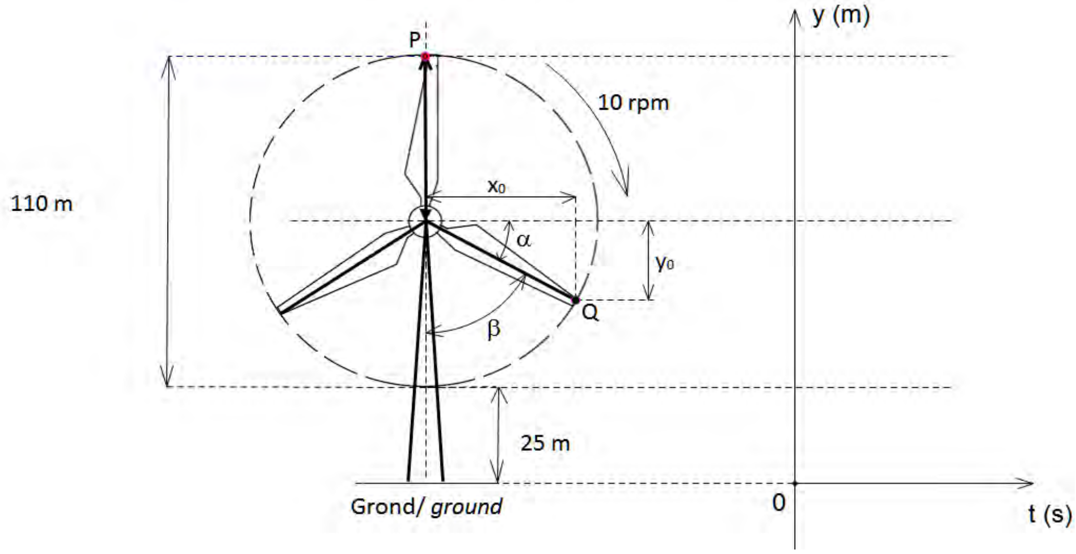
5.2. Determine the period of the movement.

5.3. Determine the maximum displacement of the cork above the ocean floor.

5.4. Sketch the graph of the movement.

6. Consider the information above which relates to a wind turbine.





Calculate the following.

6.1. Radius (length of one rotor blade)

6.2. Period (time taken to complete one full rotation cycle) in seconds

6.3. Frequency of the motion

6.4. Angular frequency

6.5. Magnitude of angle α in radians

6.6. The distances x_0 and y_0

6.7. The length of arc PQ

6.8. The time in seconds taken by one rotor blade to turn through the angle β

6.9. The phase c of the rotational motion.

6.10. Now write down a wave equation which describes y (the altitude of point Q above the ground) as a function of time. Use a sine function.

6.11. Sketch a neat graph of the function which you developed in 6.10.

5 Inequalities and absolute value

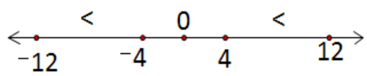
Learning aims for this study unit

Upon completion of this study unit the student must be able to do the following:

1. Solve inequalities.
2. Apply the absolute value operation to algebraic expressions.
3. Define the absolute value function as a piece-wise function.
4. Interpret the absolute value function as the "distance"-function and to represent it graphically.

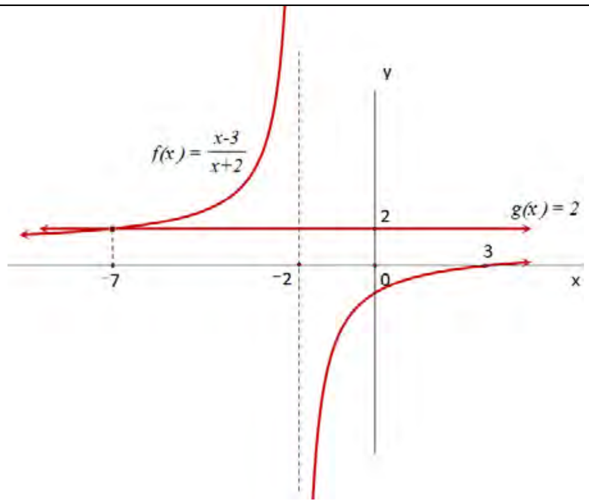
5.1 Inequalities

If we replace the $=$ in an equation with one of the relationship signs $<, \leq, >$ or $\geq$ then we obtain an inequality. Such inequalities can be solved according to a similar process as in the case of regular equations, as long as the following is kept in mind.

Property	Example
If $a < b$ then $-a > -b$. (Multiplication or division by -1 changes the sign of the relationship symbol)	$4 < 12$ is true But $-4 < -12$ is false. So $-4 > -12$ or $-12 < -4$. 
If $a < b$ then $a + c < b + c$ and $a - c < b - c$.	If $x - 3 < 0$ then $x - 3 + 3 < 0 + 3$ and y $x < 3$.
If $a < b$ and $c > 0$ then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$.	If $\frac{x}{2} < 4$ then $\frac{x}{2} \times 2 < 4 \times 2$ and $x < 8$.
If $a < b$ and $c < 0$ then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$.	If $-\frac{x}{5} < 6$ then $-\frac{x}{5} \times (-5) < 6 \times (-5)$ $x > -30$.
If $a < b$ and $b < c$ then $a < c$.	If $a < x$ and $x < 2$ then $a < 2$.

The most lethal danger associated with the solution of inequalities probably occurs in the case of rational inequalities, which CANNOT be treated the same way as rational equations. Study the following examples with great concentration.

Rational equation	**Rational inequality**
$\frac{x-3}{x+2} = 2$ $\therefore \frac{x-3}{x+2} \times \frac{x+2}{1} = 2 \times \frac{x+2}{1}$ $\therefore x - 3 = 2x + 4$ $\therefore -x = 7$ $\therefore x = -7$ THE LCM IS USED IN ORDER TO DO AWAY WITH THE FRACTION FORM Graphically:	$\frac{x-3}{x+2} > 2$ $\therefore \frac{x-3}{x+2} - 2 > 0$ $\therefore \frac{x-3}{x+2} - 2 \times \frac{x+2}{x+2} > 0$ $\therefore \frac{x-3}{x+2} - \frac{2x-4}{x+2} > 0$ $\therefore \frac{x-3-2x-4}{x+2} > 0$ BEHOU BREUKVORM/ KEEP FRACTION FORM! $\therefore \frac{-x-7}{x+2} > 0$ $\therefore \frac{x+7}{x+2} < 0$ so/so $\frac{x+7}{x+2}$ is negatief/negative We now set up intervals from the points $x = -7$ and $x = -2$:



interval	$(-\infty;-7)$ of/or $x<-7$	$(-7;-2)$ of/or $-7<x<-2$	$(-2;\infty)$ of/or $x>-2$
monsterpunt/ *sample point*	-8	-5	-1
waarde van/ *value of* $\frac{x+7}{x+2}$	0,167	−0,667	6

$\frac{x+7}{x+2}$ is negative on $-7<x<-2$.

The solution of $\frac{x-3}{x+2}>2$ is therefore $-7<x<-2$, also written as $(-7;-2)$ (with the graph above left)

Rational equation

$$\frac{2x+1}{x-1}-\frac{2}{x-3}=1$$

$$\therefore \frac{(x-1)(x-3)}{1}\times\left(\frac{2x+1}{x-1}-\frac{2}{x-3}\right)=1\times\frac{(x-1)(x-3)}{1}$$

$$\therefore (2x+1)(x-3)-2(x-1)=(x-1)(x-3)$$

$$\therefore 2x^2-7x-1=x^2-4x+3$$

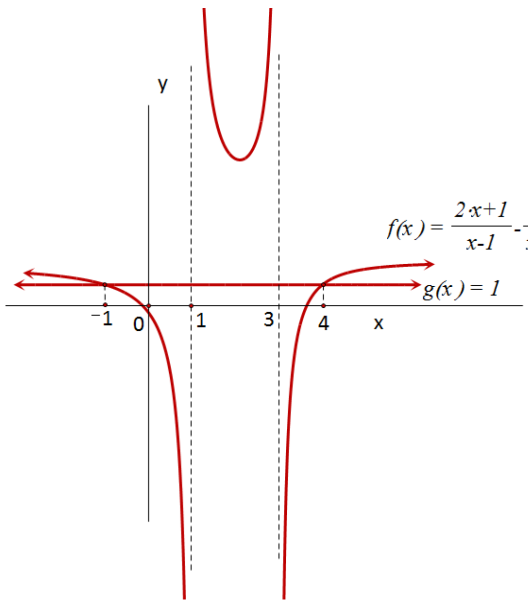
$$\therefore x^2-3x-4=0$$

$$\therefore (x-4)(x+1)=0$$

$$\therefore x=4 \text{ of / or } x=-1$$

THE LCM IS USED IN ORDER TO DO AWAY WITH THE FRACTION FORM

Graphically



Rational inequality

$$\frac{2x+1}{x-1}-\frac{2}{x-3}<1$$

$$\therefore \frac{2x+1}{x-1}-\frac{2}{x-3}-1<0$$

$$\therefore \frac{(2x+1)(x-3)}{(x-1)(x-3)}-\frac{2(x-1)}{(x-3)(x-1)}-\frac{(x-3)(x-1)}{(x-3)(x-1)}<0$$

$$\therefore \frac{x^2-3x-4}{(x-1)(x-3)}<0$$

$$\therefore \frac{(x-4)(x+1)}{(x-1)(x-3)}<0$$

We now set up intervals from the points $x=4$, $x=-1$, $x=1$ and $x=3$:

interval	$-\infty<x<-1$ of/or $(-\infty;-1)$	$-1<x<1$ of/or $(-1;1)$	$1<x<3$ of/or $(1;3)$	$3<x<4$ of/or $(3;4)$	$x>4$ of/or $(4;\infty)$
monsterpunt/ *sample point*	−2	0	2	3,5	5
waarde van/ *value of* $\frac{(x-4)(x+1)}{(x-1)(x-3)}$	0,4	−1,333	6	−1,8	0,75

$\frac{(x-4)(x+1)}{(x-1)(x-3)}$ is negative on $-1<x<1$ or $3<x<4$.

The solution is therefore $(-1,1) \cup (3,4)$ (compare with the graph left).

Exercise 5.1

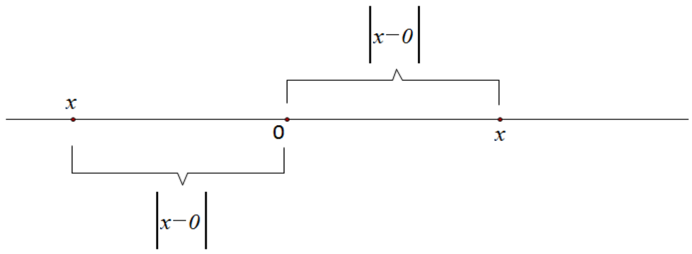
Solve the following inequalities.

1. $\frac{x-3}{x+1} \geq 0$
2. $\frac{2x+1}{x-5} \leq 3$
3. $\frac{1+x}{1-x} - \frac{1-x}{1+x} \leq -1$

5.2 Absolute value

The distance definition

$|x|$ means $|x-0|$ and that means the distance between 0 on the number line and the point x.



It is clear that x may be located on either side of 0; the distance $|x-0|$ is always positive – so, the value of $|x|$ for any value of x will always be positive. The number $|x-0|$ may be obtained by setting the radius of a compass on $|x|$ and putting the sharp pin in the zero point. Then the point x may be marked off on both sides of the zero point. The expression $|x-a|$ can geometrically be considered as **the distance** between the two number points x and a on the number line. Because distance is always a positive quantity, the expression $|x-a|$ means the same as $|a-x|$.

The formal algebraic definition

$$|x| = \begin{cases} x, & x \geq 0 \\ -x & x < 0 \end{cases} \text{ and in general } |x-a| = \begin{cases} x-a, & x-a \geq 0 \\ -(x-a) & x-a < 0 \end{cases} = \begin{cases} x-a, & x \geq a \\ a-x & x < a \end{cases}$$

The formal algebraic definition provides us with a way to easily sketch the absolute value function, defined by

$$f(x) = a|bx-d| + h$$

The secret is to treat the function as a piece-wise defined function; then the absolute value function behaves like a combination of two constrained straight line graphs. The vertex is the point on the graph about which the graph is symmetrical.

$$f(x) = a|bx-d| + h$$

$$\therefore f(x) = \begin{cases} a(bx-d)+h & \text{as/if} \quad bx-d \geq 0 \\ -a(bx-d)+h & \text{as/if} \quad bx-d < 0 \end{cases}$$

$$= \begin{cases} abx-ad+h & \text{as/if} \quad bx \geq d \\ -abx+ad+h & \text{as/if} \quad bx < d \end{cases}$$

$$= \begin{cases} \overbrace{(ab)}^{m}x + \overbrace{(-ad+h)}^{c} & \text{as/if} \quad x \geq \frac{d}{b} \\ \left(\underbrace{-ab}_{m}\right)x + \underbrace{(ac+h)}_{c} & \text{as/if} \quad x < \frac{d}{b} \end{cases}$$

The vertex is therefore at $x = \frac{d}{b}$. Substitute $x = \frac{d}{b}$ in $y = (ab)x + (-ad+h)$ in order to obtain the y coordinate.

$$y = (ab)\frac{d}{b} + (-ad+h) = h$$

The vertex is therefore at $\left(\frac{d}{b}, h\right)$.

Example

Sketch graphs of the following functions.

1. $y = 2\left|\frac{1}{4}x - 2\right| + 4$ 2. $y = -\frac{1}{4}|3x + 5| - 3$

1. $$y = 2\left|\frac{1}{4}x - 2\right| + 4$$

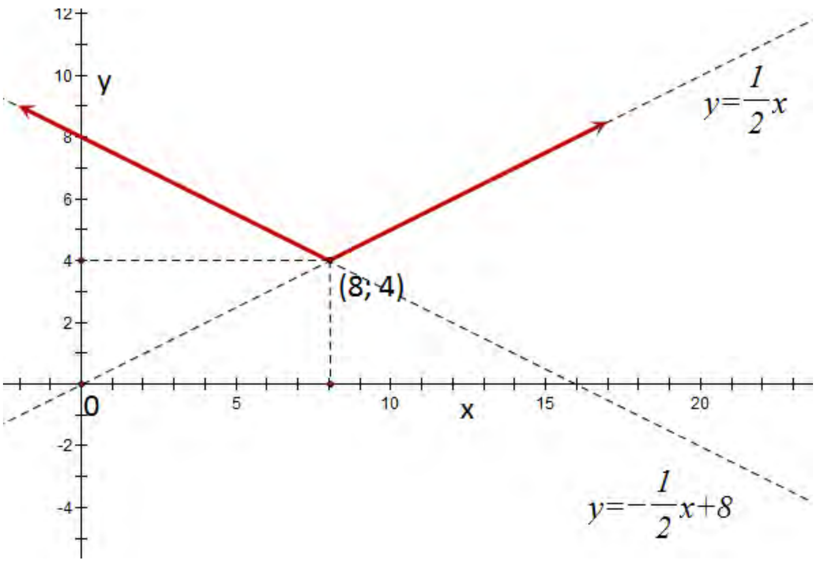
$$\therefore y = \begin{cases} 2\left(\frac{1}{4}x - 2\right) + 4 & \text{as/ if } \frac{1}{4}x - 2 \geq 0 \\ -2\left(\frac{1}{4}x - 2\right) + 4 & \text{as/ if } \frac{1}{4}x - 2 < 0 \end{cases}$$

$$= \begin{cases} \frac{1}{2}x - 4 + 4 & \text{as/ if } \frac{1}{4}x \geq 2 \\ -\frac{1}{2}x + 4 + 4 & \text{as/ if } \frac{1}{4}x < 2 \end{cases}$$

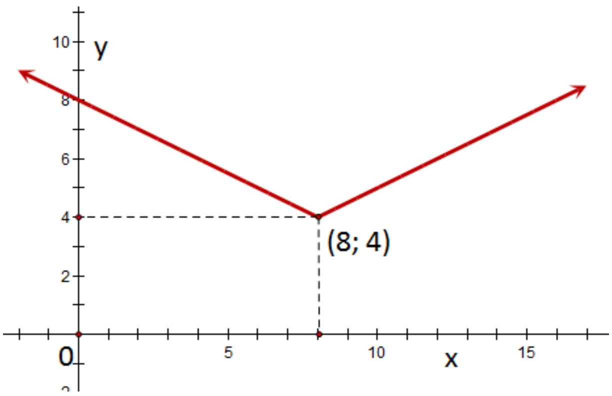
$$= \begin{cases} \frac{1}{2}x & \text{as/ if } x \geq 8 \\ -\frac{1}{2}x + 8 & \text{as/ if } x < 8 \end{cases}$$

Knakpunt/ *vertex*: $(8; 4)$

This yields



Normally, we do not show the two "auxilliary lines".



2. $$y = -\frac{1}{4}|3x+5| - 3$$

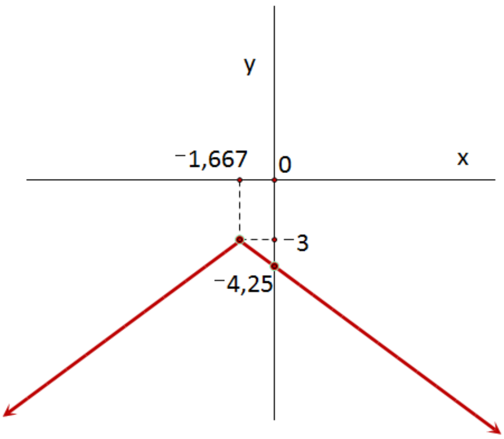
$$\therefore y = \begin{cases} -\frac{1}{4}(3x+5) - 3 & \text{as/ if} \quad 3x+5 \geq 0 \\ +\frac{1}{4}(3x+5) - 3 & \text{as/ if} \quad 3x+5 < 0 \end{cases}$$

$$= \begin{cases} -\frac{3}{4}x - \frac{5}{4} - 3 & \text{as/ if} \quad 3x \geq -5 \\ \frac{3}{4}x + \frac{5}{4} - 3 & \text{as/ if} \quad 3x < -5 \end{cases}$$

$$= \begin{cases} -\frac{3}{4}x - \frac{17}{4} & \text{as/ if} \quad x \geq -\frac{5}{3} \\ \frac{3}{4}x - \frac{7}{4} & \text{as/ if} \quad x < -\frac{5}{3} \end{cases}$$

Knakpunt/ *vertex*: $\left(-\frac{5}{3}; -3\right)$

This yields



Geometrical definition

How can we use the number line to given meaning to $|x-a| = r$? What is the geometrical meaning of $|x-a| = r$?

Since $|x-a| = r$ is the same as $|(x-a)-0| = r$, it means that $x-a$ is **precise** r units from 0 on the number line. Mark the points r and $-r$ on the number line. This is where $x-a$ is. From this it follows that $x-a=r$ or $x-a=-r$ and $x=a+r$ or $x=a-r$. In terms of distance, x is either a distance r units to the left or to the right from a. We can also state that $x = a \pm r$.

If $|x-a| < r$ it means that $x-a$ is **less** than r units from 0 on the number line. On the number line $x-a$ is therefore in the area between $-r$ and r. From this it follows that $x-a<r$ and $x-a>-r$, therefore $-r < x-a < r$ and $a-r < x < a+r$. In terms of distance, x is in the area between r units to the left and to the right from a. We can also state that $x \in (a-r, a+r)$.

If $|x-a| > r$ it means that $x-a$ is **more** than r units from 0 on the number line. On the number line $x-a$ is therefore in the area outside $-r$ and r. From this it follows that $x-a>r$ or $x-a<-r$, therefore $x > a+r$ or $x < a-r$. In terms of distance, x is in the area outside r units to the left and to the right from a. We can also state that $x \in (-\infty, a-r) \cup (a+r, \infty)$.

For $|x-a| \leq r$ it holds that $a-r \leq x \leq a+r$ or $x \in [a-r, a+r]$ and for $|x-a| \geq r$ it holds that $x \geq a+r$ or $x \leq a-r$ or $x \in (-\infty, a-r] \cup [a+r, \infty)$.

Other important properties of the absolute value operation

$$|x| = \sqrt{x^2}, -|x| = -\sqrt{x^2}, |a||b| = |ab|, \frac{|a|}{|b|} = \left|\frac{a}{b}\right|, |a-x| = |x-a|$$

Example 1

Determine x such that $2|3x-4|=20$.

Solution (3 methods):

- Geometric definition

$$2|3x-4|=20$$
$$\therefore |3x-4|=10$$
$$\therefore |(3x-4)-0|=10$$

Find the value of x such that $3x-4$ lies precisely 10 units from the point 0 on the number line.

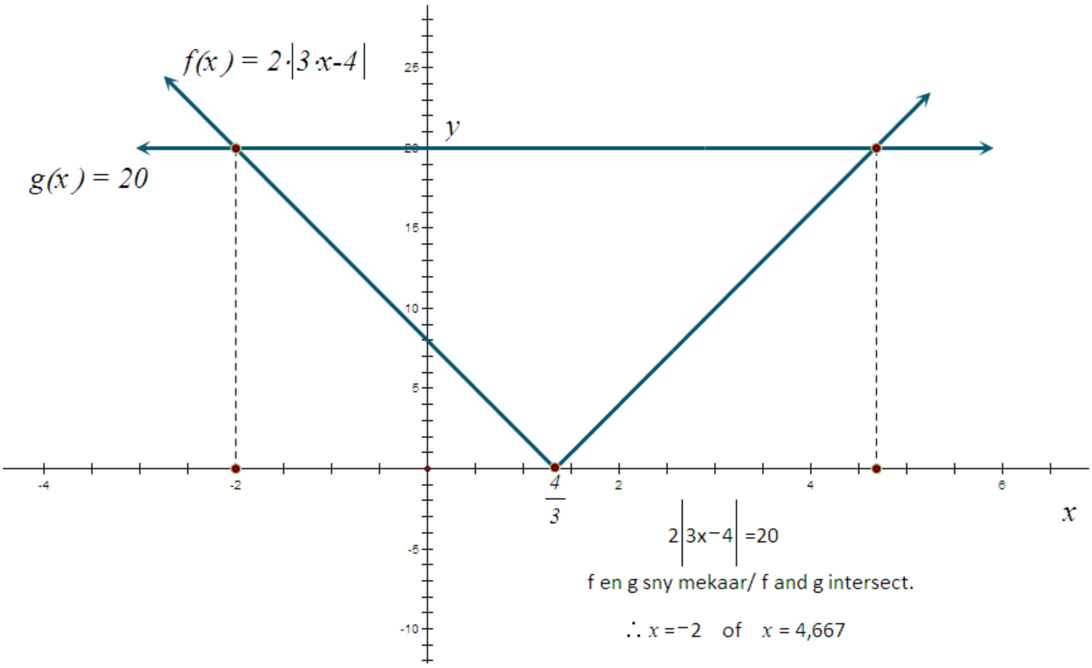
$\therefore 3x-4=-10$ of / *or* $3x-4=10$

$\therefore x=-2$ of / *or* $x=\frac{14}{3}$

- Formal definition

$$2|3x-4|=20$$
$$\therefore |3x-4|=10$$
$$\therefore \begin{cases} 3x-4=10 & \text{as } 3x-4\geq 0 \\ -(3x-4)=20 & \text{as } 3x-4<0 \end{cases}$$
$$\therefore \begin{cases} 3x=14 & \text{as } 3x\geq 4 \\ -3x+4=10 & \text{as } 3x<4 \end{cases}$$
$$\therefore \begin{cases} x=\frac{14}{3} & \text{as } x\geq \frac{4}{3} \\ -3x=6 & \text{as } x<\frac{4}{3} \end{cases}$$
$$\therefore \begin{cases} x=4,667 & \text{as } x\geq \frac{4}{3} \\ x=-2 & \text{as } x<\frac{4}{3} \end{cases}$$

- Graphical interpretation



Example 2

Solve for x such that $3|2x - 7| \leq 30$.

Solution (3 methods):

- Geometric definition

$$3|2x - 7| \leq 30$$
$$\therefore |2x - 7| \leq 10$$
$$\therefore |(2x - 7) - 0| \leq 10$$

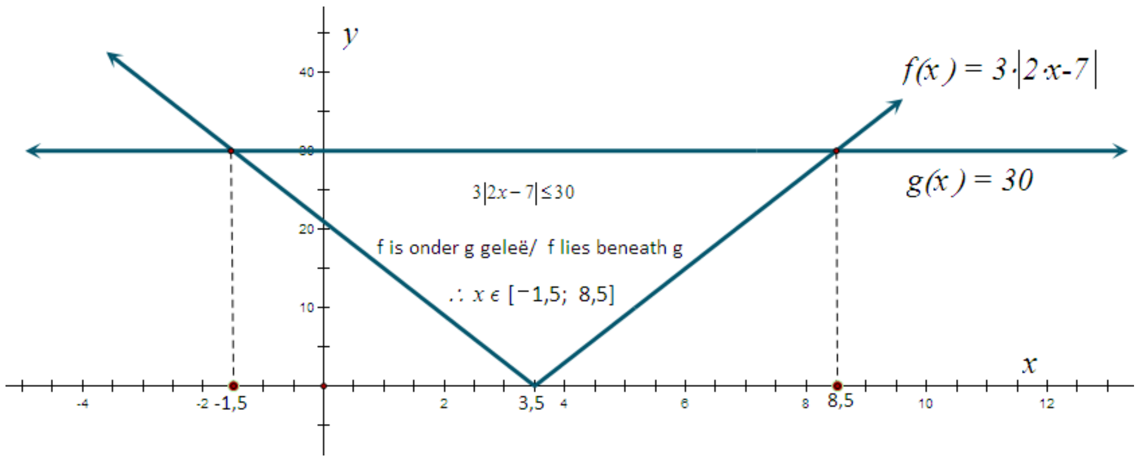
Find the value of x such that $2x - 7$ lies less than 10 units from the point 0 on the number line.

$$\therefore -10 \leq 2x - 7 \leq 10$$
$$\therefore -3 \leq 2x \leq 17$$
$$\therefore -\frac{3}{2} \leq x \leq \frac{17}{2}$$
$$\therefore x \in [-1{,}5; 8{,}5]$$

- Formal definition

$$3|2x-7| \leq 30$$
$$\therefore \begin{cases} 3(2x-7) \leq 30 & \text{as} \quad 2x-7 \geq 0 \\ -3(2x-7) \leq 30 & \text{as} \quad 2x-7 < 0 \end{cases}$$
$$\therefore \begin{cases} 6x-21 \leq 30 & \text{as} \quad 2x \geq 7 \\ -6x+21 \leq 30 & \text{as} \quad 2x < 7 \end{cases}$$
$$\therefore \begin{cases} 6x \leq 51 & \text{as} \quad x \geq \frac{7}{2} \\ -6x \leq 9 & \text{as} \quad x < \frac{7}{2} \end{cases}$$

- Graphical interpretation



Example 3

Solve for x such that $-2|3x+4| < -6$.

Solution (3 methods):

- Geometric definition

$$-2\left|3x+4\right|<-6$$
$$\therefore \left|3x+4\right|>3$$
$$\therefore \left|(3x+4)-0\right|>3$$

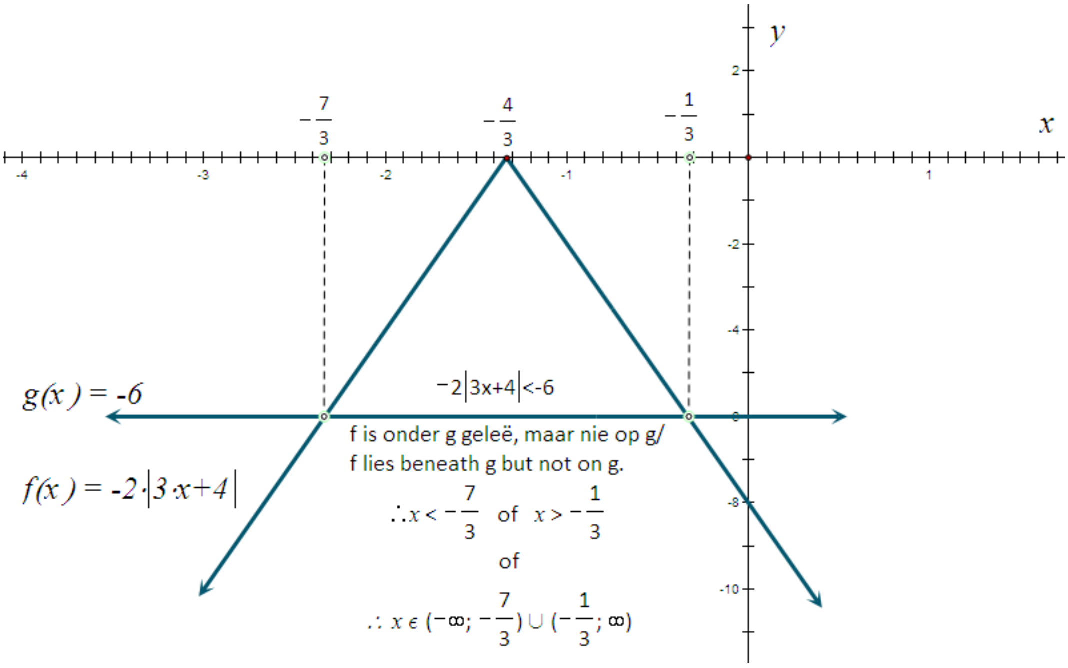
Find the value of x such that $3x + 4$ lies more than 3 units from the point 0 on the number line.

$$\therefore 3x+4<-3 \quad \text{of / } or \quad 3x+4>3$$
$$\therefore x<-\frac{7}{3} \quad \text{of / } or \quad x>-\frac{1}{3}$$
$$\therefore \left(-\infty;-\frac{7}{3}\right)\cup\left(-\frac{1}{3};\infty\right)$$

- Formal definition

$$-2\left|3x+4\right|<-6$$
$$\therefore \left|3x+4\right|>3$$
$$\therefore \begin{cases} 3x+4>3 & \text{as} \quad 3x+4\geq 0 \\ -(3x+4)>3 & \text{as} \quad 3x+4<0 \end{cases}$$
$$\therefore \begin{cases} 3x>-1 & \text{as} \quad 3x\geq -4 \\ -3x-4>3 & \text{as} \quad 3x<-4 \end{cases}$$
$$\therefore \begin{cases} 3x>-1 & \text{as} \quad x\geq -\frac{4}{3} \\ -3x>7 & \text{as} \quad x<-\frac{4}{3} \end{cases}$$

- Graphical interpretation



Exercise 5.2

- Solve for the unknown.

  .1. $3|2-k|=0$

  .2. $-2|2t+1|=6$

  .3. $\left|\frac{r}{2}+\frac{1}{2}\right|=|-2|$

- Solve for the unknown and represent the solutions on a number line.

  .1. $|x+2|\leq 2$

  .2. $-3|2x-5|<-9$

  .3. $-6\left|\frac{2-3x}{4}\right|>6$

- Write the following in absolute value notation.

  .1. x is less than 3 units from 7.

  .2. t is no more than 5 units from 8.

  .3. y is located between -3 and 3.

  .4. The distance between 6 and m is 4.

6 Limits and continuity

Learning aims for this study unit

Upon completion of this study unit the student must be able to do the following:

1. Determine the limit of a function in terms of left-sided limits and right-sided limits in the vicinity of a point.
2. Give judgment regarding the continuity of a function in a particular point.
3. Compute certain limits.
4. Apply the epsilon-delta definition of the limit of a function in order to prove that the limit of a function in a particular point exists.

6.1 Limits and continuity

Often, we are interested in the value which is assumed by a function when the independent variable becomes a very large negative value, or a very large positive value, or when the independent variable assumes a particular value. It is not always possible to simply substitute the independent variable into the function equation and then obtain the function value. Next, we consider functions of which the behaviour can only be fully described in terms of limits.

Example 1

Consider the function.

$$f(x) = \frac{2x+1}{x-1} - \frac{2}{x-3}$$

1. Determine $f(2)$, that is the function value in the point where $x = 2$.
2. The function value in the point where $x = 2$ provides no information about the behaviour of the function close to the point where $x = 2$. Let us now investigate the behaviour of the function in the vicinity or proximity of this point. Complete the following table.

As/*if* $x<2$		As/*if* $x>2$	
x	$f(x)=\frac{2x+1}{x-1}-\frac{2}{x-3}$	x	$f(x)=\frac{2x+1}{x-1}-\frac{2}{x-3}$
1		2,001	
1,5		2,01	
1,9		2,1	
1,99		2,5	
1,999		3	

2.1. To what does the function value tend when x tends to 2 from the left-hand side?

$\lim_{x\to 2^-}\left(\frac{2x+1}{x-1}-\frac{2}{x-3}\right)=$..................... or shorter $\lim_{x\to 2^-} f(x)=$.....................

2.2. To what does the function value tend when x tends to 2 from the right-hand side?

$\lim_{x\to 2^+} f(x)=$.....................

2.3. How do the answers from 2.1 and 2.2 compare? $\lim_{x\to 2^-} f(x)$ .... $\lim_{x\to 2^+} f(x)$

When the left-sided limit and the right-sided limit of a function assumes the same value in a point, then we say that the function has a limit in that point.

Because $\lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)=$............. we can write $\lim_{x\to 2} f(x)=$.............

The limit of a function in a particular point only exists provided that both the left-sided limit and the right-sided limit exist in that point and that they are equal in value. In this case we have that both the function value in the point where $x = 2$ and the limit of the function when x tends from both sides to 2 yield the same value.

$$\lim_{x \to 2} f(x) = f(2)$$

This special set of circumstances implies that f is **continuous** (solid curve without gaps or breaks or jumps) in the point where $x = 2$.

3. Let us now investigate the behaviour of the function where x becomes a very large negative value and where x becomes a very large positive value. Complete the following table.

As/if $x \to -\infty$		As/if $x \to \infty$	
x	$f(x) = \frac{2x+1}{x-1} - \frac{2}{x-3}$	x	$f(x) = \frac{2x+1}{x-1} - \frac{2}{x-3}$
−10		10	
−100		100	
−1000		1000	
−10 000		10 000	

3.1. To what does the function value tend when x tends to negative infinity?

$\lim_{x \to -\infty} \left(\frac{2x+1}{x-1} - \frac{2}{x-3} \right) = \ldots\ldots\ldots\ldots\ldots\ldots$ or shorter $\lim_{x \to -\infty} f(x) = \ldots\ldots\ldots\ldots\ldots\ldots$

3.2. To what does the function value tend when x tends to positive infinity?

$\lim_{x \to \infty} f(x) = \ldots\ldots\ldots\ldots\ldots\ldots$

From the limits $\lim_{x \to -\infty} f(x)$ and $\lim_{x \to \infty} f(x)$ we can often find the **horizontal asymptotes** of a function.

4. Determine $f(3)$, that is the function value in the point where $x = 3$.

5. The function value in the point where $x = 3$ provides no information about the behaviour of the function close to the point where $x = 3$. Let us now investigate the behaviour of the function in the vicinity or proximity of this point. Complete the following table.

As/if $x < 3$		As/if $x > 3$	
x	$f(x) = \frac{2x+1}{x-1} - \frac{2}{x-3}$	x	$f(x) = \frac{2x+1}{x-1} - \frac{2}{x-3}$
2		3,001	
2,5		3,01	
2,9		3,1	
2,99		3,5	
2,999		4	

5.1. To what does the function value tend when x tends to 3 from the left-hand side?

$\lim_{x \to 3^-} \left(\frac{2x+1}{x-1} - \frac{2}{x-3} \right) = \ldots\ldots\ldots\ldots\ldots\ldots$ or shorter $\lim_{x \to 3^-} f(x) = \ldots\ldots\ldots\ldots\ldots\ldots$

5.2. To what does the function value tend when x tends to 3 from the right-hand side?

$\lim_{x \to 3^+} f(x) = \ldots\ldots\ldots\ldots\ldots\ldots$

5.3. How do the answers from 5.1 and 5.2 compare?

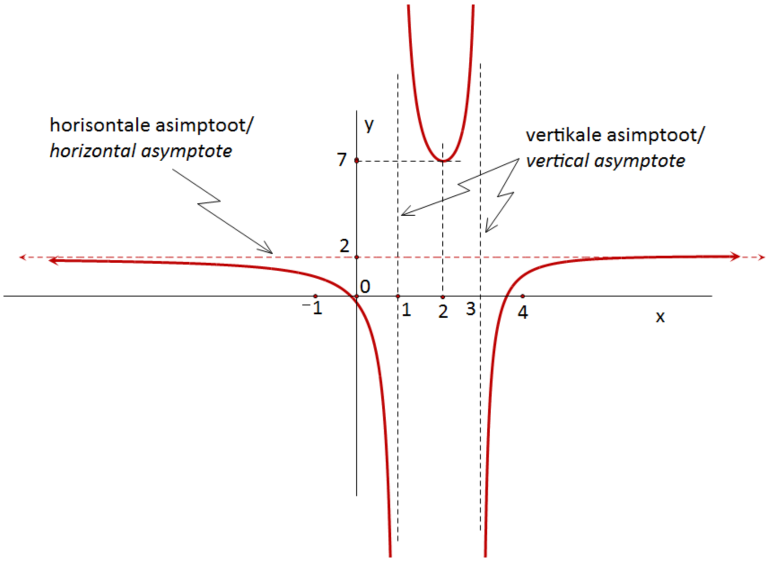
$\lim_{x \to 3^-} f(x)$ .... $\lim_{x \to 3^+} f(x)$

When the left-sided limit and the right-sided limit of a function does not assume the same value in a point, then we say that the function does not have a limit in that point.

Because $\lim_{x \to 3^-} f(x) \neq \lim_{x \to 3^+} f(x)$ we can write that $\lim_{x \to 3} f(x)$ does not exist.

Because the left-sided limit and the right-sided limit each does not tend to a fixed real value, neither the left-sided limit nor the right-sided limit exists in this case. We refer to the vertical line with equation $x = 3$ as a **vertical asymptote** of the function. The limit of a function in a particular point only exists provided that the left-sided limit and the right-sided limit both exist in that point and that they have the same value. In this case we have that the function value in the point where $x = 3$ does not exist and that the limit of the function when x tends to 3 from both sides does not exist. This special set of circumstances implies that the function f is **discontinuous** (not solid, with a gap or a jump) in the point where $x = 3$.

For clarity I show a computer-generated representation of the function we have investigated.



Example 2

Consider the function.

$$f(x) = \begin{cases} x^2, & x < 1 \\ 2, & x = 1 \\ x, & x > 1 \end{cases}$$

1. Determine $f(1)$, that is the function value in the point where $x = 1$.
2. The function value in the point where $x = 1$provides no information about the behaviour of the function close to the point where $x = 1$. Let us now investigate the behaviour of the function in the vicinity or proximity of this point. Complete the following table.

As/*if* $x < 1$		As/*if* $x > 1$	
x	$f(x) = \begin{cases} x^2 & \text{as/if } x < 1 \\ 2 & \text{as/if } x = 1 \\ x & \text{as/if } x > 1 \end{cases}$	x	$f(x) = \begin{cases} x^2 & \text{as/if } x < 1 \\ 2 & \text{as/if } x = 1 \\ x & \text{as/if } x > 1 \end{cases}$
0		1,001	
0,5		1,01	
0,9		1,1	
0,99		1,5	
0,999		2	

2.1. To what does the function value tend when x tends to 1 from the left-hand side?

$\lim_{x \to 1^-} f(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

2.2. To what does the function value tend when x tends to 2 from the right-hand side?

$\lim_{x \to 1^+} f(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

2.3. How do the answers from 2.1 and 2.2 compare? $\lim_{x \to 1^-} f(x) \ldots\ldots \lim_{x \to 1^+} f(x)$

When the left-sided limit and the right-sided limit of a function assumes the same value in a point, then we say that the function has a limit in that point.

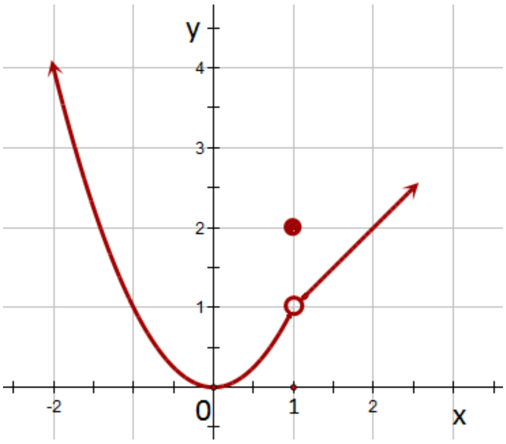
Because $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = \ldots\ldots\ldots\ldots$ we can write $\lim_{x \to 1} f(x) = \ldots\ldots\ldots\ldots$

The limit of a function in a particular point only exists provided that both the left-sided limit and the right-sided limit exist in that point and that they are equal in value. In this case we have that the function value in the point where $x = 1$ and the limit of the function when x tends from both sides to 1 yield the different values.

$\lim_{x \to 1} f(x) \neq f(1)$

This special set of circumstances implies that f is **discontinuous** in the point where $x = 1$.

For clarity I show a computer-generated representation of the function we have investigated.



Example 3

Consider the function.

$$f(x) = \frac{8 - x^3}{2 - x}$$

1. Determine $f(2)$, that is the function value in the point where $x = 2$.
2. The function value in the point where $x = 2$ provides no information about the behaviour of the function close to the point where $x = 2$. Let us now investigate the behaviour of the function in the vicinity or proximity of this point. Complete the following table.

As/*if* $x < 2$		As/*if* $x > 2$	
x	$f(x) = \frac{8-x^3}{2-x}$	x	$f(x) = \frac{8-x^3}{2-x}$
1		2,001	
1,5		2,01	
1,9		2,1	
1,99		2,5	
1,999		3	

2.1. To what does the function value tend when x tends to 2 from the left-hand side?

$\lim_{x\to 2^-} f(x) = ...$

2.2. To what does the function value tend when x tends to 2 from the right-hand side?

$\lim_{x\to 2^+} f(x) = ...$

2.3. How do the answers from 2.1 and 2.2 compare? $\lim_{x\to 2^-} f(x)$ .... $\lim_{x\to 2^+} f(x)$

When the left-sided limit and the right-sided limit of a function assumes the same value in a point, then we say that the function has a limit in that point.

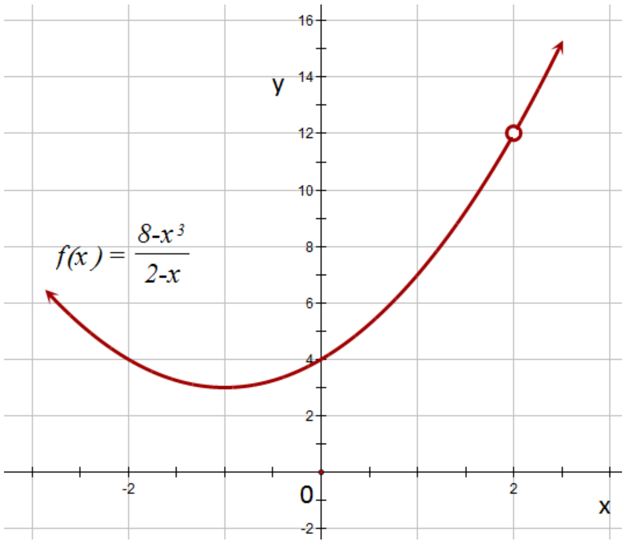
Because $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) =$ .... we can write $\lim_{x\to 2} f(x) =$ ....

The limit of a function in a particular point only exists provided that both the left-sided limit and the right-sided limit exist in that point and that they are equal in value. **NOTE that the function need not be defined in that point. The limit exists whenever both the left-sided limit and the right-sided are equal in value.** In this case we have that the function value in the point where $x = 2$ and the limit of the function when x tends from both sides to 2 do not yield the same value. **In fact, the function value does not exist.**

$\lim_{x\to 2} f(x) \neq f(2)$

This special set of circumstances implies that f is **discontinuous** in the point where $x = 2$.

For clarity I show a computer-generated representation of the function we have investigated.



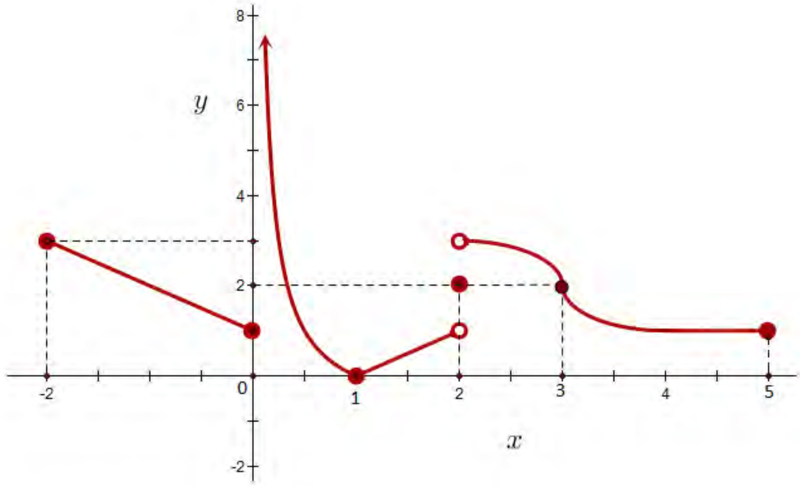
Note, however, that the limit of this function may also be obtained in the following way.

$$\lim_{x\to 2} \frac{8-x^3}{2-x}$$
$$= \lim_{x\to 2} \frac{(2-x)(4+2x+x^2)}{2-x}$$
$$= \lim_{x\to 2} \frac{\cancel{(2-x)}(4+2x+x^2)}{\cancel{(2-x)}}$$
$$= \lim_{x\to 2} (4+2x+x^2)$$
$$= 4+2(2)+2^2$$
$$= 12$$

This type of limit, **where the numerator which would become zero during direct substitution may be removed by factorization and division**, is called a **removable discontinuity**. They are characterized by the form $\frac{0}{0}$ which appears when the value which the independent variable is approaching, is substituted directly into the function. The division operation here is permissible, because the denominator only tends to zero – it does not actually "become" zero.

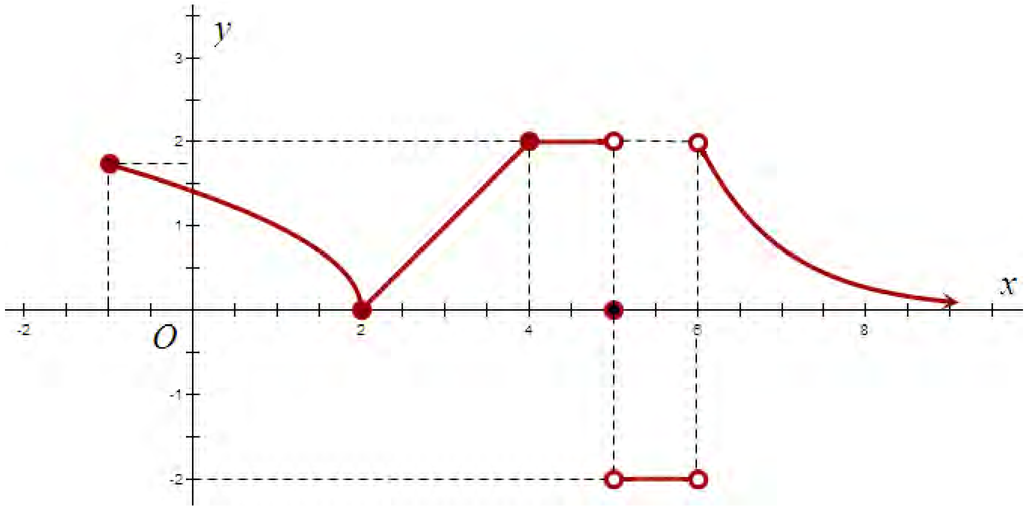
Exercise 6.1

1. Consider the piecewise-defined function f.



List four points where f is discontinuous. Only list the x-coordinates of the points <u>as well</u> as the reason why f is not continuous in that point.

2. Consider the piecewise-defined function g.



List three points where g is discontinuous. Only list the x-coordinates of the points <u>as well</u> as the reason why g is not continuous in that point.

3. Use the conditions for continuity and show that the following function is continuous in the specified point.

$$g(t) = \frac{t-3}{9t}, t = -3$$

6.2 Calculation of certain limits

Direct method for calculating limits boil down to avoiding a table method but attempting instead direct substitution. When direct substitution fails we test whether or not we are dealing with one of the following special cases.

The special case $\frac{0}{0}$

We encountered this type at the end of Study Section 6.1.

Example

Calculate the limit

$$\lim_{p \to -5} \frac{p^2 - 25}{p + 5}$$

Solution

$$\lim_{p\to -5}\frac{p^2-25}{p+5} \qquad \text{vervanging lewer / } \textit{substitution yield} \quad \frac{(-5)^2-25}{-5+5}=\frac{0}{0}$$

$$\begin{aligned}
&= \lim_{p\to -5}\frac{(p-5)\cancel{(p+5)}}{\cancel{(p+5)}}\\
&= \lim_{p\to -5}(p-5)\\
&= -5-5\\
&= -10
\end{aligned}$$

The special case $\frac{\infty}{\infty}$

When dealing with this type, we divide each term by the highest power of the variable which occurs.

Example

Calculate the limit.

$$\lim_{t\to\infty}\frac{t^2}{t^2-1}$$

Solution

$$\lim_{t\to\infty}\frac{t^2}{t^2-1} \qquad \text{vervanging lewer / } \textit{substitution yield} \quad \frac{\infty}{\infty}$$

$$\begin{aligned}
&= \lim_{t\to\infty}\frac{\left(\frac{t^2}{t^2}\right)}{\left(\frac{t^2}{t^2}-\frac{1}{t^2}\right)}\\
&= \lim_{t\to\infty}\frac{1}{\left(1-\frac{1}{t^2}\right)} \qquad \text{maar/ } \textit{but } \frac{1}{t^n}\to 0 \text{ as / } \textit{if } \; t\to\infty \text{ en / } \textit{and } n>0\\
&= \frac{1}{(1-0)}\\
&= 1
\end{aligned}$$

Exercise 6.2

Calculate the limits directly (no table).

1. $\lim_{x\to 0}\frac{10x^2}{x}$
2. $\lim_{x\to -1}\frac{x^2-1}{x+1}$
3. $\lim_{t\to 3}\frac{t^2-5t+6}{t-3}$
4. $\lim_{k\to -4}\frac{64+k^3}{4+k}$
5. $\lim_{x\to\infty}\frac{10x^2}{x}$
6. $\lim_{x\to\infty}\frac{3x^2-5x+7}{2x^2-x}$
7. $\lim_{t\to\infty}\frac{2t+5}{3t+2}$
8. $\lim_{m\to\infty}\frac{m^3-27}{m-3}$

7 Introduction to function analysis

Previously acquired knowledge from Study Unit 3 which is required for this study unit:

From the knowledge you acquired at school or from the revision we did in Study Unit 3, you should already be able to do the following:

1. Apply the formal definition of a function as a special relation
2. Identify the domain and range of a function
3. Determine the inverse of a given function
4. Perform operations with functions

You are welcome to page back if needed.

Learning aims for this study unit

Upon completion of this study unit the student must be able to do the following:

1. Calculate the derivative of a function by using the definition of a derivative in terms of the limit of the slope of the secant through a curve when the distance between the points of intersection becomes infinitesimal:

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

2. Determine whether or not a function is differentiable in a point or not, by using the slope of the tangent to a curve
3. Apply differentiation rules in order to calculate the derivative of certain simple functions
4. Calculate the derivative of composite functions
5. Apply differentiation in order to sketch and analyze the graphs a variety of functions

7.1 The derivative of a function from first principles

The slope of the secant intersecting a curve as the average rate of change of a function

Suppose a function is defined as $f(x) = 3x^2 + 45$. Consider two points A and B on the curve of the function such that A is at $x = 3$ and B is at $x = 6$.

1. Calculate the coordinates of A and B.
2. Now calculate the slope of the secant AB.

Note that the slope of the secant AB actually measures "how fast" the function value changes over the interval $[3,6]$. Because the interval is long, we refer to the slope of the secant as the "average rate of change of the function values with respect to the independent variable". That is what we mean by the symbol

$$m_{AB} = \frac{\Delta y}{\Delta x}.$$

The slope of the tangent to a curve as the instantaneous rate of change of a function

Suppose a function is defined as $f(x) = 3x^2 + 45$. Consider two points A and B on the curve of the function such that A is at $x = 3$ and B is at $x = 3 + h$ to the right of A.

1. Calculate the coordinates of A and B.
2. Now calculate the slope of the secant AB in terms of h.

Now suppose we make the value of h very small, so that the point B approaches the point A. Then the secant AB will begin to resemble a tangent to the curve at A.

$$m_{AB} \to m_{\text{tangent at A}} \text{ if } h \to 0$$

So, we may formulate the movement of point B to A in order to change the secant AB into a tangent, as

$$m_{\text{tangent at A}} = \lim_{h \to 0} m_{AB}.$$

That is, in terms of our usual notation

$$\begin{aligned} m_{\text{tangent at } A} &= \lim_{h\to 0}\frac{\Delta y}{\Delta x} \\ &= \lim_{h\to 0}\frac{y_B - y_A}{x_B - x_A} \\ &= \lim_{h\to 0}\frac{f(3+h)-f(3)}{3+h-3} \end{aligned}$$

Note that you have just calculated the value of $\frac{f(3+h)-f(3)}{3+h-3}$ in terms of h. So, all you need to do now in order to obtain the slope of the tangent at A, is to let the value of h become infinitesimal (infinitely small). The value which you obtained gives the slope of the tangent to the curve at the point A, that is where $x = 3$. Because the interval over which the slope of the tangent has been calculated, that is the magnitude of h, is infinitesimal we should note that the slope of the tangent measures the instantaneous rate of change of the function value with respect to the independent variable. We indicate the instantaneous rate of change, that is the slope of the tangent, with the symbol $\frac{dy}{dx}$ and we refer to it as the derivative of the function.

Example

Calculate the derivative of the function $f(t) = 4t^3 - 2t^2$ from first principles.

Solution

Consider points $A(t, 4t^3 - 2t^2)$ and $B(t + h, 4(t + h)^3 - 2(t + h)^2)$ on the curve of f.

The two points are therefore

$$A(t, 4t^3 - 2t^2)$$

and

$$B(t + h, 4t^3 + 12ht^2 + 12h^{2t} + 4h^3 - 2t^2 - 4ht - 2h^2).$$

$$\begin{aligned} \text{Slope of secant } AB \quad &= \frac{\Delta f}{\Delta t} \\ &= \frac{f_2 - f_1}{t_2 - t_1} \\ &= \frac{4t^3 + 12ht^2 + 12h^2t + 4h^3 - 2t^2 - 4ht - 2h^2 - \left(4t^3 - 2t^2\right)}{t + h - t} \\ &= \frac{4t^3 + 12ht^2 + 12h^2t + 4h^3 - 2t^2 - 4ht - 2h^2 - 4t^3 + 2t^2}{h} \\ &= \frac{12ht^2 + 12h^2t + 4h^3 - 4ht - 2h^2}{h} \\ &= \frac{h\left(12t^2 + 12ht + 4h^2 - 4t - 2h\right)}{h} \\ &= 12t^2 + 12ht + 4h^2 - 4t - 2h \end{aligned}$$

Laat B na A beweeg deur h baie klein te maak/ *Let B move to A by making h very small* :

$$\begin{aligned} \text{Slope of tangent at A} \quad &= \lim_{h\to 0}\left(\text{ Slope of secant } AB\right) \\ &= \lim_{h\to 0}\left(12t^2 + 12ht + 4h^2 - 4t - 2h\right) \\ &= 12t^2 + 0 + 0 - 4t - 0 \\ &= 12t^2 - 4t \end{aligned}$$

Note that some mathematicians prefer to present the calculation of a derivative in a more compact form.

Example

Calculate the derivative of the function $y = \sqrt{y}$ from first principles.

Solution

$$
\begin{aligned}
\frac{dy}{dx} &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h} \\
&= \lim_{h\to 0}\left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\times 1\right) \\
&= \lim_{h\to 0}\left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\times \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right) \\
&= \lim_{h\to 0}\frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)} \\
&= \lim_{h\to 0}\frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)} \\
&= \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} \\
&= \frac{1}{\sqrt{x+0}+\sqrt{x}} \\
\therefore \frac{dy}{dx} &= \frac{1}{2\sqrt{x}}
\end{aligned}
$$

Exercise 7.1

Calculate the derivatives of the following functions from first principles.

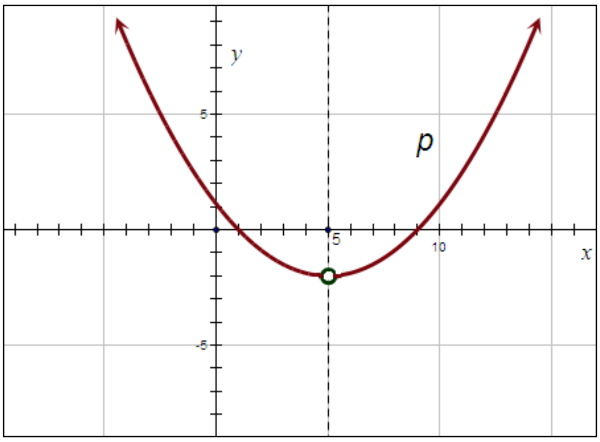
1. $g(t) = \frac{3}{\sqrt{t}}$
2. $s(t) = \frac{2t-3}{3t+2}$
3. $f(x) = 3x^2 - 3x + 4$
4. $f(x) = \sin x$. Use $\lim_{h\to 0}\frac{\sin h}{h} = 1$ and $\lim_{h\to 0}\frac{\cos h - 1}{h} = 0$.
5. $f(x) = \cos x$. Use $\lim_{h\to 0}\frac{\sin h}{h} = 1$ and $\lim_{h\to 0}\frac{\cos h - 1}{h} = 0$.
6. $g(p) = |2p|$

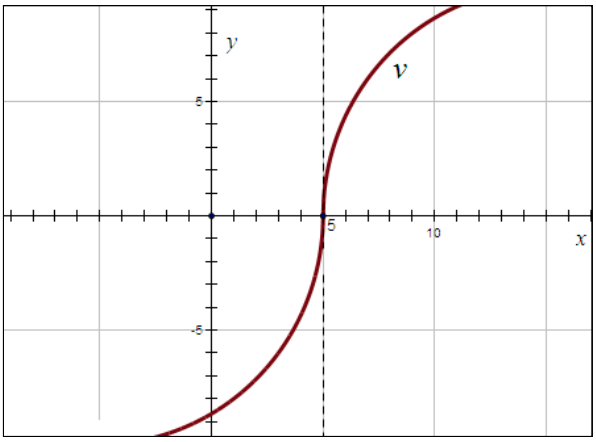
7.2 Differentiability

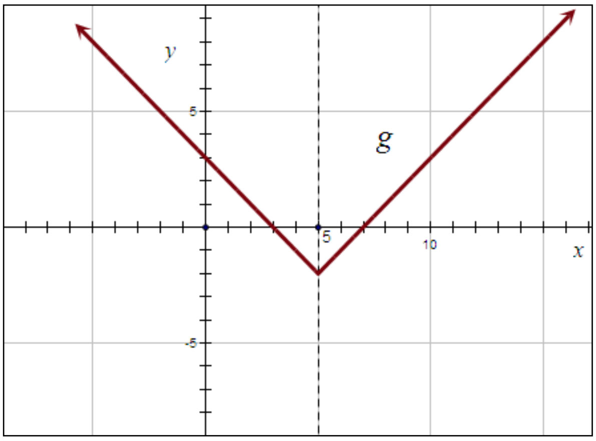
From the previous Study Section follows that the concept of a derivative is inseparably part of the concept of a tangent. Therefore, we may consider differentiability as a property which means that if a function is differentiable on an open interval, then it is at any point in that open interval possible to draw a unique tangent with real-valued slope to the curve of the function. In principle this comes down to the fact that a function is differentiable where it is continuous and smooth. So, 'n function is non-differentiable where it has a sharp point, where it is discontinuous or where it runs vertically.

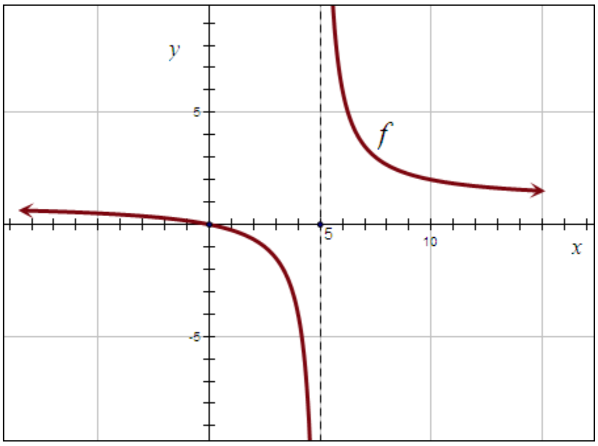
Exercise 7.2

Investigate all the conditions for differentiability and specify which of the conditions do not hold at each of the following cases.

1. 

2. 

3. 

4. 

7.3 Differentiation rules

In Study Section 7.1 we experienced that differentiation from first principles is a long and tedious process. Fortunately, convenient differentiation rules may easily be derived by applying the definition of a derivative to a variety of functions. You shall prove or derive some of these differentiation rules later in the course of your studies; for today we shall assume their validity without proof.

Note that the notation D_x means $\frac{dy}{dx} = f'(x)$.

Constants: $D_x(\text{constant}) = 0$

Addition: $D_x[f(x) + g(x)] = D_x f(x) + D_x g(x)$

Subtraction: $D_x[f(x) + g(x)] = D_x f(x) + D_x g(x)$

Scalar multiplication: $D_x[c\, f(x)] = c\, D_x f(x)$

Power function: $D_x x^n = n\, x^{n-1}, n \in \mathbb{R}$

Product rule: $D_x[f(x)g(x)] = g(x)D_x f(x) + f(x)D_x g(x)$

Quotient rule: $D_x \frac{f(x)}{g(x)} = \frac{g(x)D_x f(x) - f(x)D_x g(x)}{[g(x)]^2}$, $g(x) \neq 0$.

Chain rule: $\frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{du}\frac{du}{dx}$ met $y = f(v)$, $v = g(u)$ en $u = h(x)$ of $D_v f(v) D_u g(u) D_x h(x)$

Trigonometric functions: $D_x \sin x = \cos x$, $D_x \cos x = -\sin x$, $D_x \tan x = \sec^2 x$, $D_x \csc x = -\csc x \cot x$, $D_x \sec x = \sec x \tan x$, $D_x \cot x = -\csc^2 x$

Exponential functions: $D_x e^x = e^x$, $D_x a^x = a^x \ln a$

Logarithmic functions: $D_x \ln x = \frac{1}{x}$, $D_x \log_a x = \frac{1}{x \ln a}$

Absolute value function: $D_x|x| = \frac{x}{|x|}$

Let us now investigate their use.

Exercise 7.3

Calculate the derivatives in each of the following cases by making use of differentiation rules.

1. Calculate the average slope of $f(x) = x^2 + 2$ between $x = 3$ and $x = 5$.
2. If $f(x) = 3x^2 + 2$, determine the slope of the tangent to the curve f at the point $x = -2$.
3. Differentiate with respect to x.

   3.1. $f(x) = 4x^3 + x\sqrt{x} - \frac{5}{x^4}$

   3.2. $f(x) = (x^2 + 6x)(x^3 - 6x^2)$ in two different ways.

   3.3. $f(x) = \frac{x^2+3}{5}$

   3.4. $f(x) = \frac{x^2-1}{x^2+1}$

7.4 Composite functions and the chain rule

First revise Study section 3.5.

The chain rule is given as $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ where $y = f(u)$ and $u = g(x)$ or $(f \circ g)'(x) = f'\big(g(x)\big)g'(x)$.

Example

$h(x) = \cos(x^2 - x + 1)$ with $y = \cos u$ and $u = x^2 - x + 1$

so that $\frac{dh}{dx} = \frac{dy}{du}\frac{du}{dx} = -\sin x \cdot (2x - 1) = -\sin(x^2 - x + 1) \cdot (2x - 1)$.

Or $h(x) = \cos(x^2 - x + 1)$ with $f\big(g(x)\big) = \cos\big(g(x)\big)$ and $g(x) = x^2 - x + 1$

So that $h'(x) = f'\big(g(x)\big)g'(x) = -\sin(x^2 - x + 1) \cdot (2x - 1)$.

Exercise 7.4

1. Determine the derivatives of the following functions.

   1.1. $f(x) = \sqrt{3x-2}$

   1.2. $g(x) = \frac{1}{(x^2-1)^2}$

   1.3. $h(x) = (x^3 - x^2 + x + 1)^5$

   1.4. $j(x) = \sqrt[4]{4x^2 - 6x}$

   1.5. $k(x) = (2x^2 + 4x)^{100}$

   1.6. $m(x) = \cos^2 x$

2. The volume of a cube is given by the formula $V(x) = x^3$. Suppose the side length x of the cube, measured in in mm, decreases according to the formula $x(t) = \frac{3}{10}t^2 - 6t + 30$ where t is measured in minutes. Determine the rate of change of the volume with time, measured in mm^3/min, when $t = 5$ minutes.

3. The area A of an oil slick which is floating on the surface of the sea due to a damaged tank ship is approximated by a circle, which implies that $A(r) = \pi r^2$. The radius $r(t) = -\frac{t^2}{5} + 4t$, $0 \leq t \leq 10$, with r in km and t in hours, of the circle increases as the oil spreads so that the radius is 20 km after 10 hours.

   3.1. Calculate the rate at which the radius is growing after 5 hours.

   3.2. Calculate the area of the oil slick at the time when the radius is increasing at 1.2 km/h.

7.5 Application of differentiation

Analysis (the application of calculus to functions) is one of the most powerful tools ever developed by mankind. Although you have only encountered differentiation calculus at this stage, you are already able to mathematically handle a large variety of problems which involve rates of change. Soon you will also encounter integration.

Exercise 7.5

1. The mathematics of demand and supply

   The demand for a merchant's product is given by the function $q = 8000 - 40p$, with q the number of units per week demanded by the consumers if the price per unit is p rand. The cost function for the manufacturing of the product is given by $C = 100000 + 20q$.

   1.1. Determine the price and quantity at the break-even points, i.e. the point(s) where no profit is made.

   1.2. Determine the level of production which will yield a maximum revenue.

   1.3. Determine the price per unit which will yield a maximum revenue.

   1.4. What is the maximum revenue?

   1.5. Determine the level of production which will yield a maximum profit.

   1.6. Determine the price per unit which will yield a maximum profit.

   1.7. What is the maximum profit?

2. Financial risk

   An economist must analyse a prospective business opportunity. The person proposes that the expected profit (P) in rands as a function of time (t) in months is given by $P = 10000 + 30t^2 - \frac{2}{3}t^3$.

   2.1. In which month(s), if any, will the rate of growth in profit be zero?

   2.2. During which month will the rate of growth in profit be the fastest?

   2.3. What is the highest rate of growth that the business will achieve

   2.4. What is the highest profit for any month that the business will achieve?

3. Rate of change

The temperature of a mixture in a factory, changes according to the function $T(t) = -\frac{t^2}{4} + \frac{5t}{2} + \frac{375}{4}$ for $0 \leq t \leq 25$ with T in °C and t in minutes.

3.1. Differentiate $T(t)$ using differentiation rules.

3.2. Calculate $\left.\frac{dT}{dt}\right|_{t=5}$ and draw a conclusion from your answer. (What is the implication of your answer?)

3.3. Calculate the temperature when the rate of temperature change is -7.5℃/minute.

4. Sketching of curves

4.1. Given $f(x) = x^3 - 4x^2 - 11x + 30$. A and B are the turning points of the given function.

4.1.1. Sketch the graphs of $f(x)$, $f'(x)$ and $f''(x)$ on the same set of axes. Indicate the coordinates of the turning points (A and B) and the point of inflection.

4.1.2. What is the meaning of the concept "inflection point"?

4.1.3. Calculate the average rate of change of the function from A to B.

4.1.4. Calculate the equation of the tangent to the curve at $x = 1$.

4.2. Given $H(x) = -x^3 + 5x^2 + 8x - 12 = (x - 1)(-x^2 + 4x + 12)$.

4.2.1. Find the coordinates of the critical points.

4.2.2. Apply the first derivative test and the second derivative test to show at which the critical points there are local minima or maxima.

4.2.3. Find the coordinate(s) of the points where the concavity changes.

4.2.4. Apply the second derivative test to show whether the concavity did change.

4.3. Given $g(x) = \frac{2x+6}{-6x+3}$.

4.3.1. Determine the vertical and horizontal asymptotes by utilizing asymptotes.

4.3.2. Determine all intercepts with axes.

4.3.3. Sketch the graph of the function. Indicate all the information that you have determined in previous two questions.

5. Optimization

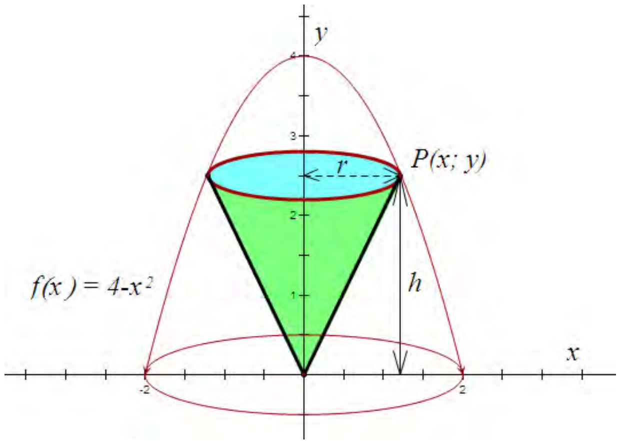
5.1. An open box is manufactured by removing four identical square sections from the corners of a sheet of cardboard which is 24 cm long and 18 cm wide and then folding the remaining parts upwards to form the sides of the container.

5.1.1. Show that the volume of the cardboard box is given by the equation $V(x) = 4x^3 - 84x^2 + 432x$ for $x \in [0,9]$.

5.1.2. Sketch a neat graph of the function V against x for $-2 \leq x \leq 14$. Clearly indicate all intercepts with axes, as well as turning points and points of inflection, in co-ordinate form.

5.1.3. What should the area of each square be in order for the volume of the box to be a maximum and what is the value of this maximum volume?

5.2. A right cone is generated by revolving the part of the parabola $y = 4 - x^2$ between the points $x = 0$ and $x = 2$ around the y axis and inscribing the cone inside the paraboloid so that its vertex is located at the origin and its base touches the parabola at the point P. Assume all units are in cm.



5.2.1. Show that the volume of the cone is given by the function $V(x) = \frac{4}{3}\pi x^3 - \frac{1}{3}\pi x^4$.

5.2.2. Sketch the curve of V against x. Take into account all constraints on the domain. Indicate all roots, turning points and inflection points within the constrained domain.

5.2.3. Employ your calculations and results from the previous question and calculate the maximum possible volume which the cone can have.

Answer to exercises

1 Logic, algebraic proficiency and exponents

Exercise 1.1

1.1. Factorise, $(2x-3)(2-x)$; Completing the square, $-2\left(x-\frac{7}{4}\right)^2+\frac{1}{8}$
1.2. Simplify, p^3-8; Determine the value if $p=1$, -7
1.3. Solve the equation, $t=\pm 2$; Determine the values for which the equation does not hold, $t\in(-\infty,-2)\cup(-2,2)\cup(2,\infty)$
1.4. Solve the equation, $k=0,\frac{7}{5},-\frac{2}{3}$; Present the solutions graphically. Sketch the function $f(k)=30k^3-22k^2-28k$ and indicate the intercepts on the horizontal axis.
1.5. Determine $f(2)$, $6\sqrt{2}+\frac{37}{2}$; Determine $f'(x)$, $\frac{3}{\sqrt{x}}+\frac{3}{x^2}+10x$
1.6. Determine the roots of $P(x)$, $x=-2,3,5$; Determine $P'(x)$, $3x^2-12x-1$
1.7. Solve the equation, $x=-2,3,5$; Determine the other factors of the expression, $x-3$ en $x-5$
1.8. Prove the identity; For which values is the identity valid? $\theta\in[90°k,90°(k+1)]$, $k=0,\pm 2,\pm 4,\ldots$
1.9. Determine the value of x, $x=2$ of $x=3$; Determine the value of α; $\alpha=36.87°$ of $\alpha=53.13°$
2. Hint: $x=2r$
3. Hint: $x^2+x^2=(2r)^2$
4. Hint: $x=2r$
5. Hint: $x^2+x^2=l^2$, $(2r)^2=x^2+l^2$

Exercise 1.2

1.1. 34
1.2. -18
1.3. -2
1.4. -90
2.1. 0
2.2. Undefined
2.3. Undefined
2.4. -21

Exercise 1.3

1.1. $4a^4+16a^3b+20a^2b^2+16ab^3+4b^3$
1.2. $-\frac{729}{8}x^6$
1.3. z^{-7}
1.4. p^3-m^3
1.5. $8x^3+27y^3$
1.6. $a^3-3a^2b+3ab^2-b^3$
1.7. $t^{-2}+t^{-3}$
1.8. $\frac{2}{5}r$
1.9. $3^{-4}=\frac{1}{81}$
1.10. $\frac{3}{2}$
2.1. $g'(t)=\frac{3}{2}t^{-\frac{1}{2}}-\frac{2}{5}t^{-2}=\frac{3}{2\sqrt{t}}-\frac{2}{5t^2}$
2.2. $g'(t)=\frac{8}{3}t^{-\frac{1}{3}}+\frac{10}{3}t^{-3}-\frac{1}{3}t^{-\frac{3}{2}}=\frac{8}{3\sqrt[3]{t}}+\frac{10}{3t^3}-\frac{1}{3\sqrt{t^3}}$
3. $20^{x-1}+10^y$

Exercise 1.4

1.1. $y=\frac{7}{8}$
1.2. $p=6\frac{1}{4}$
1.3. $x=-2$

1.4. $x = -1$, $2^x > 0$
1.5. $r = -\frac{3}{2}$, the unknown is in the base position.

Exercise 1.5

1. Use graphical software to draw the graphs, for example, desmos.
2. $3e^2 + 2 \approx 24.1672$

2 Logarithms

Exercise 2.1

1.1. 3
1.2. 0
1.3. 2
1.4. 15
1.5. 0
1.6. 6
1.7. x
1.8. -1
2.1. 81
2.2. 2
2.3. -1/3
2.4. 1

Exercise 2.2

1.1. $t = 10$
1.2. $n = 8$
1.3. $t \approx 3.2375$
1.4. $n \approx 0.08481$
2.1. For a quadratic function the difference of the difference in sequential data points is constant.
2.2. $k = \frac{\ln 4}{7} \approx 0.198$, $A = e^{-\frac{2\ln 4}{7}} \approx 0.673$
2.3. $P(46) = 6075.32$
2.4. March 2016
3.1. $k = \frac{\ln 2}{138.376} \approx \frac{1}{200} = 0.005$
3.2. $\frac{1}{k} = 200$
3.3. 2.92 kg
3.4. $200 \ln 8 \approx 415.88$ days

3 Introduction to functions

Exercise 3.1

1. Domain, horizontal, dependent variable, 8, 4, 2, 1.75, 4, 14
2. Only the data as dots on the graph.
3. Connect the dots.
4. $g(t) = t^2 - t + 2$
5. $x < 0.5$ descending, $x > 0.5$ ascending, concavity upwards, minimum turning point at $(0.5, 3.75)$
6. $D_g = \{x | -2 \leq x \leq 4; x \in \mathbb{R}\}$
7. $W_g = \{y | 1.75 \leq y \leq 14; y \in \mathbb{R}\}$
8. Extend the parabola to the left and right sides.
9. $g(-1.5) = 5.75$, $g(1.5) = 2.75$, $g(5) = 22$, $g(2 + h) = h^2 + 3h + 4$
10. Use guidelines on the graph.
11. $t = 0.5 + 0.5\sqrt{21} \approx 2.79$
12. Use guidelines on the graph.
13.1. A, C
13.2. C

14.1. $\{(-1,-1),(0,0),(1,1),(2,2),(3,3)\}$
14.2. $\{(-1,1),(0,0),(1,1),(2,4),(3,9)\}$
15. $\{(-1,-1),(0,0),(1,1),(2,2),(3,3)\}$ one-to-one function, $\{(-1,1),(0,0),(1,1),(2,4),(3,9)\}$ not one-to-one function
16. Use graphical software, for example, desmos, to verify your graphs.

Exercise 3.2

1. $s(0) = 3$, $s(1) = 3$, $s(t+h) = 3$
2. $T_1 = 3.5$, $T_4 = \frac{7}{16}$
3. $g(0)$ is undefined, $g(t) = \frac{3}{2t}$, $g\left(\frac{2}{3}\right) = 2.25$

Exercise 3.3

1. $f^{-1}(x) = \frac{x}{2} - \frac{1}{2}$, all domains and ranges are the set of real numbers.
2. $f^{-1}(x) = x^2 + 1, x \geq 0$ since $y = f(x) \geq 0$. $D_f = [1, \infty)$, $W_f = [0, \infty)$, $D_{f^{-1}} = [0, \infty)$, $W_{f^{-1}} = [1, \infty)$

Exercise 3.4

1.1. $2x^2 + 3x + 5$
1.2. $2x^2 + 3x + 5$
1.3. $2x^2 - 3x - 5$
1.4. $-2x^2 + 3x + 5$
1.5. $6x^3 + 10x^2$
1.6. $6x^3 + 10x^2$
1.7. $\frac{2x^2}{3x+5}, x \neq -\frac{5}{3}$
1.8. $\frac{3x+5}{2x^2}, x \neq 0$
2. Addition of functions and multiplication of functions
3.1. 24
3.2. 72
3.3. $a[f(x)] \neq f(ax)$
4.1. 1250
4.2. 674
4.3. $g(a+x) \neq g(a) + g(x)$
5. $\sin x$ and $\sin 4x$ have an amplitude of 1, while $4\sin x$ has an amplitude of 4. $\sin x$ and $4\sin x$ have a period of $360°$, while $\sin 4x$ has period of $90°$.

Exercise 3.5

1.1. $v\big(u(t)\big) = \sqrt[3]{\sin t}$
1.2. $u\big(v(t)\big) = \sin \sqrt[3]{t}$
2.1. $v\big(u(t)\big) = \frac{2-2t}{4-t}, t \neq 4\ (u(t) \neq -1)$
2.2. $u\big(v(t)\big) = \frac{3+3t}{t-1}, t \neq 1\ (v(t) \neq 1)$

4 Radial measure and trigonometry

Exercise 4.1

$\frac{\pi}{6}, 45, 60, \frac{\pi}{2}, \frac{2\pi}{3}, 135, \frac{5\pi}{6}, \pi, 210, \frac{5\pi}{4}, 240, \frac{3\pi}{2}, 300, \frac{7\pi}{4}, \frac{11\pi}{6}$

Exercise 4.2

1. $159°$
2. $2x$
3. $171.887339°$

Exercise 4.3

1. 10.908 m^2

2. 50% decrease (half)
3. 100% increase (double)
4. 7 m

Exercise 4.4

1.1. 3
1.2. $5+\frac{\sqrt{3}}{2}$
1.3. 2
1.4. 5
1.5. $-\frac{2}{3}$
1.6. 1
1.7. 1
1.8. 1
2.1. $LHS = \csc^2\theta = c^2/b^2$, $RHS = \cot^2\theta + 1 = \frac{a^2}{b^2}+1 = \frac{a^2+b^2}{b^2} = \frac{c^2}{b^2}$ so that $LHS = RHS$.
2.2. Similar to 2.1
2.3. Similar to 2.1
3.1. 1
3.2. $\begin{cases} 1, \cos\theta > 0 \\ -1, \cos\theta < 0 \end{cases}$
3.3. -1
4. Use graphical software, for example, desmos, to verify your graphs.
5.1. $\frac{7\pi}{6}, \frac{11\pi}{6}$
5.2. $\frac{\pi}{6}, \frac{11\pi}{6}$
5.3. $\frac{5\pi}{6}, \frac{11\pi}{6}$
5.4. $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$
5.5. $\frac{2\pi}{3}, \frac{5\pi}{6}, \frac{5\pi}{3}, \frac{11\pi}{6}$
6.1. $x = \frac{\pi}{8}+\frac{k\pi}{2}$, $x = \frac{7\pi}{12}+k\pi$, $k \in \mathbb{Z}$
6.2. $x = \frac{\pi}{2}+k\pi$, $k \in \mathbb{Z}$

Exercise 4.5

1.1. $\frac{33}{65}$
1.2. $-\frac{16}{65}$
1.3. $\frac{33}{56}$
2. $\frac{\sqrt{3}+1}{2\sqrt{2}}$
3. Proof
4. 0
5. 1

Exercise 4.6

Proving of identities

Exercise 4.7

1. 4470.6 cm^2
2. 1.1522 mile
3. 47.55 sea mile, N 84.6^o O

4.1.1. $y = \sin(2x \pm \pi) + 1$
4.1.2. $y = \sin\left[2\left(x \pm \frac{\pi}{2}\right)\right] + 1$
4.1.3. $y = \cos\left(2x+\frac{\pi}{2}\right)+1$, $y = \cos\left(2x-\frac{3\pi}{2}\right)+1$
4.1.4. $y = \cos\left[2\left(x+\frac{\pi}{4}\right)\right]+1$, $y = \cos\left[2\left(x-\frac{3\pi}{4}\right)\right]+1$

4.2. None of the two.
4.3.1. Odd
4.3.2. None of the two.
4.3.3. Even
5.1. 10 min^{-1}
5.2. 0.1 min
5.3. 8.2 m
5.4. Use graphical software, for example, desmos, to verify your graph.
6.1. 55 m
6.2. 6 s
6.3. $\frac{1}{6}$ s^{-1}
6.4. $\frac{\pi}{3}$ rad/s
6.5. $\frac{\pi}{6}$
6.6. $x_0 = \frac{55\sqrt{3}}{2}$ m, $y_0 = \frac{55}{2}$ m
6.7. $\frac{110\pi}{3}$ m
6.8. 1 s
6.9. $-\frac{\pi}{6}$
6.10. $y = -55\sin\left[\frac{\pi}{3}\left(x + \frac{1}{2}\right)\right] + 80$
6.11. Use graphical software, for example, desmos, to verify your graph.

5 Inequalities and absolute value

Exercise 5.1

1. $x < -1$ or $x \geq 3$
2. $x < 5$ or $x \geq 16$
3. $-1 < x \leq 2 - \sqrt{5}$ or $1 < x \leq 2 + \sqrt{5}$

Exercise 5.2

1.1. $k = 2$
1.2. No solution
1.3. $r = -5, r = 3$
2.1. $-4 \leq x \leq 0$
2.2. $x < 1$ or $x > 4$
2.3. $x < -\frac{2}{3}$ of $x > 2$
3.1. $|x - 7| < 3$
3.2. $|t - 8| \leq 5$
3.3. $|y| < 3$
3.4. $|6 - m| = 4$

6 Limits and continuity

Exercise 6.1

1. $x = -2$, $\lim_{x\to-2^-} f(x) \nexists$, function not defined for $x < -2$; $x = 0$, $\lim_{x\to-0^+} f(x) \nexists$ infinite discontinuity; $x = 2$, $\lim_{x\to2^-} f(x) \neq \lim_{x\to2^+} f(x)$, jump discontinuity; $x = 5$, $\lim_{x\to-5^+} f(x) \nexists$, function not defined for $x > 5$.
2. $x = -1$, $\lim_{x\to-1^-} f(x) \nexists$, function not defined for $x < -1$; $x = 5$, $\lim_{x\to5^-} f(x) \neq \lim_{x\to5^+} f(x)$, jump discontinuity; $x = 6$, $\lim_{x\to6^-} f(x) \neq \lim_{x\to6^+} f(x)$, jump discontinuity.
3. $g(-3) = \frac{2}{9} = \lim_{t\to-3} \frac{t-3}{9t}$

Exercise 6.2

1. 0
2. -2

3. 1
4. 48
5. ∞
6. $\frac{3}{2}$
7. $\frac{2}{3}$
8. ∞

7 Introduction to function analysis

Exercise 7.1

1. $-\frac{3}{2t\sqrt{t}}$
2. $\frac{13}{(3t+2)^2}$
3. $6x-3$
4. $\cos x$
5. $-\sin x$
6. $\frac{2p}{|p|}$

Exercise 7.2

1. Discontinuous
2. Runs vertically
3. Sharp point / salient point
4. Discontinuous

Exercise 7.3

1. 8
2. -12

3.1. $12x^2+\frac{3}{2}\sqrt{x}+\frac{20}{x^5}$

3.2. $(2x+6)(x^3-6x^2)+(x^2+6x)(3x^2-12x)=5x^4-108x^2$ en $\frac{d}{dx}(x^5-36x^3)=5x^4-108x^2$

3.3. $\frac{2x}{5}$

3.4. $\frac{2x(x^2+1)-(x^2-1)2x}{(x^2-1)^2}=\frac{4x}{(x^2-1)^2}$

Exercise 7.4

1.1. $f'(x)=\frac{3}{2\sqrt{3x-2}}$

1.2. $g'(x)=-\frac{4x}{(x^2-1)^3}$

1.3. $h'(x)=5(x^3-x^2+x+1)^4(3x^2-2x+1)$

1.4. $j'(x)=\frac{1}{4}(8x-6)(4x^2-6x)^{-\frac{3}{4}}=\frac{4x-3}{2\sqrt[4]{(4x^2-6x)^3}}$

1.5. $k'(x)=400(2x^2+4x)^{99}(x+1)$

1.6. $m'(x)=-2\cos x\sin x$

2. -506.25 mm^3/min

3.1. 2 km/h

3.2. 7 hours, 1040.62 km^2

Exercise 7.5

1.1. 607 items at R184.83 per item and 6593 items at R35.17

1.2. 4000

1.3. R100

1.4. R400000

1.5. 3600

1.6. R110
1.7. R224000
2.1. In the beginning and the 30st month
2.2. 15th month
2.3. R450/ month
2.4. R19000 in the 30st month
3.1. $T'(t) = -\frac{t}{2} + \frac{5}{2}$
3.2. 0. Temperature changes from increase to decrease.
3.3. $43.75°C$
4.1.1. Use graphical software, for example, desmos, to verify your graph. Turning points at $A = (-1,36)$ and $B = \left(\frac{11}{3}, -\frac{400}{27}\right)$. Inflection point at $\left(\frac{3}{4}, \frac{286}{27}\right)$.
4.1.2. Point where the concavity changes.
4.1.3. $-\frac{98}{9}$
4.1.4. $y = -16x + 32$
4.2.1. $\left(-\frac{2}{3}, -\frac{400}{27}\right), (4,36)$
4.2.2. $H\left(-\frac{2}{3}\right) = -\frac{400}{27}$ is a local minimum and $(4,36)$ is a local maximum.
4.2.3. $\left(\frac{5}{3}, \frac{286}{27}\right)$
4.2.4. $H''(x) > 0, x < \frac{5}{3}$ and $H''(x) < 0, x > \frac{5}{3}$.
4.3.1. Vertical asymptote at $x = \frac{1}{2}$ and horizontal asymptote at $y = -\frac{1}{3}$.
4.3.2. y-intercept at $x = -3$ and x-intercept at $y = 2$.
4.3.3. Use graphical software, for example, desmos, to verify your graph.
5.1.1. $V(x) = (24 - 2x)(18 - 2x)x$
5.1.2. Use graphical software, for example, desmos, to verify your graph. Turning points at $(7 - \sqrt{13}, 655)$ and $(7 + \sqrt{13}, -95)$. Inflection point at $(7,280)$.
5.1.3. 11.5 cm^2, 655 cm^3
5.2.1. Hint: $V = \frac{\pi}{3} r^3 h$ with $r = x$ and $h = y$.
5.2.2. Use graphical software, for example, desmos, to verify your graph. $x \in [0,2]$ vir $V(x) \geq 0$. Turning points at $(0,0)$ and $\left(\sqrt{2}, \frac{4\pi}{3}\right)$. Inflection point at $\left(\sqrt{\frac{2}{3}}, \frac{20\pi}{27}\right)$.
5.2.3. $V(\sqrt{2}) = \frac{4\pi}{3}$

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